18.781 Problem Set 2 - Fall 2008

Due Tuesday, Sep. 23 at 1:00

1. (Niven 1.3.39) Prove that

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2007} - \frac{1}{2008} = \frac{1}{1005} + \frac{1}{1006} + \cdots + \frac{1}{2008}.$$

   You may find it easier to prove a general statement!

2. (Niven 1.3.4, 1.3.5, and 1.3.8) Write $n = a_m a_{m-1} \ldots a_1 a_0$ in decimal digits, so that $n = a_m 10^m + a_{m-1} 10^{m-1} + \cdots + a_1 10 + a_0$.

   (a) Prove that n is divisible by 3 if and only if $a_m + a_{m-1} + \cdots + a_0$ is divisible by 3.

   (b) Prove that n is divisible by 9 if and only if $a_m + a_{m-1} + \cdots + a_0$ is divisible by 9.

   (c) Prove that n is divisible by 11 if and only if

$$a_m - a_{m-1} + a_{m-2} - \cdots + (-1)^{m-1} a_1 + (-1)^m a_0$$

   is divisible by 11.

   (d) Prove that n is divisible by 7 if and only if $n' - 2a_0$ is divisible by 7, where $n' = (n - a_0)/10$. Explain how this can be iterated to give a divisibility test for 7 and use it on $n = 39333$.

   (Bonus) For any prime $p > 5$, use the fact that there exists a solution to $xp \equiv 1 \pmod{10}$ to devise a divisibility test.

3. (Niven 1.3.10 and 1.3.26)

   (a) Prove that any number of the form $3k + 2$ has a prime factor of the same form. Do the same for numbers of the form $4k + 3$ and $6k + 5$.

   (b) Prove that there are infinitely many primes of the form $3k + 2, 4k + 3$, and $6k + 5$.

4. (Niven 1.3.21) Prove that for positive integers a, b, c,

$$[a, b, c](ab, bc, ca) = abc.$$

5. (a) Prove that there is not unique factorization in the set $\{a + b\sqrt{-7} \mid a, b \in \mathbb{Z}\}$.

   (b) If D is an odd, positive integer that is not a square number, prove that there is not unique factorization in $\{a + b\sqrt{-D} \mid a, b \in \mathbb{Z}\}$.

   (Bonus) Consider the *Gaussian integers*, which is the set of "integer coordinate complexes": $\{a + bi \mid a, b \in \mathbb{Z}\}$. Prove that in this set, $p = 1 + i$ has the following property of prime numbers:
   If $p \mid (a + bi)(c + di)$, then either $p \mid (a + bi)$ or $p \mid (c + di)$.

6. (Niven 2.1.7) Show that if $f(x)$ is a polynomial with integral coefficients and $f(a) \equiv k$ mod m, then $f(a + tm) \equiv k \pmod{m}$ for any t.

7. (Niven 2.1.20) Prove that $42 \mid (n^7 - n)$ for any n.

8. (Niven 2.1.25) Prove that $91 \mid (n^{12} - a^{12})$ for any a, n that are both coprime to 91. Give an counterexample showing that this condition is necessary.

9. (Niven 2.1.27) Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for any n.

10. (Niven 2.2.2) Let $N(k)$ denote the number of solutions to $f(x) \equiv k \pmod{m}$. Prove using a simple counting argument that

$$\sum_{k=1}^{m} N(k) = m.$$

11. (Niven 2.2.5acd) Find *all* solutions of the congruences

    (a) $20x \equiv 4 \pmod{30}$,

    (b) $353x \equiv 254 \pmod{400}$,

    (c) $57x \equiv 87 \pmod{105}$.

(Bonus) *Postage stamp problem.* Since $(5, 12) = 1$, we know that the linear combination $5x + 12y = n$ can be solved for all n. However, consider the related problem of characterizing positive linear combinations, with $x, y \geq 0$ (this situation would arise if we had only 5 and 12 cent stamps). In that case, we cannot solve the cases $n = 1, 2, 3, 4, 6, \ldots$.

  (a) Prove that there is some bound N such that $5x + 12y = n$ has solutions whenever $n \geq N$. List all of the n for which there is no solution - how many such n are there?

  (b) Repeat part (a) for general $(a, b) = 1$: find a bound N such that $ax + by = n$ has solutions for $n \geq N$, and characterize and count the n for which there is no solution.