18.781 Problem Set 9 - Fall 2008

Due Thursday, Nov. 13 at 1:00

1. (Niven 7.8.2) Suppose that N is a nonzero integer. Prove that if $x^2 - dy^2 = N$ has one solution, then it has infinitely many.

2. (Niven 7.8.3) Prove that $x^2 - dy^2 = -1$ has no solution if $d \equiv 3 \pmod 4$.

3. (Niven 7.1.1) Expand the following fractions into simple continued fractions:

   (a) $\frac{17}{3}$

   (b) $\frac{3}{17}$

   (c) $\frac{8}{1}$

   (d) $\frac{71}{34}$.

4. Prove that if $x = [a_0, a_1, \ldots, a_r]$ is greater than 1, then $\frac{1}{x} = [0, a_0, a_1, \ldots, a_r]$.

5. (Niven 7.1.3) Convert the continued fractions into rational numbers:

   (a) $[2, 1, 4]$

   (b) $[-3, 2, 12]$

   (c) $[0, 1, 1, 100]$.

6. (Niven 7.1.4 & 7.1.5) Suppose that $c > d$, and that all a_i are integers.

   (a) Prove that $[a_0, c] < [a_0, d]$.

   (b) Prove that $[a_0, a_1, c] > [a_0, a_1, d]$.

   (c) Prove that $[a_0, a_1, \ldots, a_r, c] < [a_0, a_1, \ldots, a_r, d]$ if and only if r is even, with the opposite (strict) inequality when r is odd.

7. (Niven 7.3.1 & 7.3.2)

   (a) Evaluate $[1, 1, 1, \ldots]$.

   (b) Evaluate $[2, 1, 1, 1, \ldots]$.

   (c) Evaluate $[2, 3, 1, 1, 1, \ldots]$.

8. (Niven 7.3.4) For $i \geq 1$, prove that

$$\frac{k_i}{k_{i-1}} = [a_i, a_{i-1}, \ldots, a_2, a_1].$$

   Find and prove a similar formula for $\frac{h_i}{h_{i-1}}$ (*Hint: Use the Euclidean algorithm on k_i and k_{i-1}*).

9. Calculate the first three convergents for

   (a) e^2

   (b) 2π.

10. Calculate the infinite continued fraction expansions for

(a) $\sqrt{7}$

(b) $\frac{1+\sqrt{13}}{2}$.

(Bonus) Let n be a positive integer.

(a) Prove that $\sqrt{n^2+1} = [n, \overline{2n}]$.

(b) Prove that $\sqrt{n^2+2} = [n, \overline{n, 2n}]$.

(c) Prove that $\sqrt{n^2+2n} = [n, \overline{1, 2n}]$.