18.781 Solutions to Problem Set 9 - Fall 2008
Due Thursday, Nov. 13 at 1:00

1. (Niven 7.8.2) Suppose that N is a nonzero integer. Prove that if $x^2 - dy^2 = N$ has one solution, then it has infinitely many.

   We are given that $x^2 - dy^2 = N$ has one solution, let's call it (x_0, y_0). Then we know that there are infinitely many solutions (x_i, y_i) to Pell's equation:

   $$x_i^2 - dy_i^2 = 1.$$

   For each of these solutions, we consider the following:

   $$\begin{aligned}
   N = N \cdot 1 &= (x_0^2 - dy_0^2)(x_i^2 - dy_i^2) \\
   &= (x_0 - \sqrt{d}y_0)(x_0 + \sqrt{d}y_0)(x_i - \sqrt{d}y_i)(x_i + \sqrt{d}y_i) \\
   &= (x_0 - \sqrt{d}y_0)(x_i - \sqrt{d}y_i)(x_0 + \sqrt{d}y_0)(x_i + \sqrt{d}y_i) \\
   &= \Big((x_0x_i + dy_0y_i) - \sqrt{d}(x_0y_i + y_0x_i)\Big)\Big((x_0x_i + dy_0y_i) + \sqrt{d}(x_0y_i + y_ox_i)\Big) \\
   &= (x_0x_i + dy_0y_i)^2 - d(x_0y_i + y_0x_i)^2.
   \end{aligned}$$

   So each of the infinite solutions to Pell's equation gives a solution to the given equation, so there are infinitely many.

2. (Niven 7.8.3) Prove that $x^2 - dy^2 = -1$ has no solution if $d \equiv 3 \pmod 4$.

   If $d \equiv 3 \pmod 4$, then looking at the equation mod 4, we get the congruence

   $$x^2 + y^2 \equiv 3 \pmod 4.$$

   But the squares mod 4 are just 0 and 1. Two of these clearly cannot be combined to make 3, so there are no solutions to the equation for these values of d.

3. (Niven 7.1.1) Expand the following fractions into simple continued fractions:

   (a) $\frac{17}{3}$
   (b) $\frac{3}{17}$
   (c) $\frac{8}{1}$
   (d) $\frac{71}{34}$.

   (a)

   $$\frac{17}{3} = 5 + \frac{2}{3} = 5 + \frac{1}{\frac{3}{2}} = 5 + \cfrac{1}{1 + \cfrac{1}{2}} = [5, 1, 2].$$

   (b)

   $$\frac{3}{17} = 0 + \frac{1}{\frac{17}{3}} = 0 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}} = [0, 5, 1, 2]$$

(c)

$$\frac{8}{1} = 8 = [8].$$

(d)

$$\frac{71}{34} = 2 + \frac{3}{34} = 2 + \frac{1}{\frac{34}{3}} = 2 + \cfrac{1}{11 + \cfrac{1}{3}} = [2, 11, 3].$$

4. Prove that if $x = [a_0, a_1, \ldots, a_r]$ is greater than 1, then $\frac{1}{x} = [0, a_0, a_1, \ldots, a_r]$.
   If $x > 1$, then $a_0 > 0$, since it is equal to $\lfloor x \rfloor$. Therefore,

$$\frac{1}{x} = \frac{1}{[a_0, a_1, \ldots, a_r]} = 0 + \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}} = [0, a_0, a_1, \ldots, a_r].$$

5. (Niven 7.1.3) Convert the continued fractions into rational numbers:

   (a) $[2, 1, 4]$
   (b) $[-3, 2, 12]$
   (c) $[0, 1, 1, 100]$.

(a)

$$2 + \cfrac{1}{1 + \cfrac{1}{4}} = 2 + \frac{1}{\frac{5}{4}} = 2 + \frac{4}{5} = \frac{14}{5}.$$

(b)

$$-3 + \cfrac{1}{2 + \cfrac{1}{12}} = -3 + \frac{1}{\frac{25}{12}} = -3 + \frac{12}{25} = -\frac{63}{25}.$$

(c)

$$0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{100}}} = \cfrac{1}{1 + \cfrac{1}{\frac{101}{100}}} = \cfrac{1}{1 + \cfrac{100}{101}} = \frac{1}{\frac{201}{101}} = \frac{101}{201}.$$

6. (Niven 7.1.4 & 7.1.5) Suppose that $c > d$, and that all a_i are integers.

   (a) Prove that $[a_0, c] < [a_0, d]$.
   We have $c > d$, so $\frac{1}{c} < \frac{1}{d}$, and by adding a_0 to both sides we get the inequality.

   (b) Prove that $[a_0, a_1, c] > [a_0, a_1, d]$.
   Let $c' = [a_1, c]$ and $d' = [a_1, d]$. Then by the above argument, we get $d' > c'$, and so, using the above argument again, we get the middle inequality in the following:

$$[a_0, a_1, c] = [a_0, c'] > [a_0, d'] = [a_0, a_1, d].$$

(c) Prove that $[a_0, a_1, \ldots, a_r, c] < [a_0, a_1, \ldots, a_r, d]$ if and only if r is even, with the opposite (strict) inequality when r is odd.

So this is an induction on r. Above we have shown that the statement holds for $r = 0, 1$. Now suppose the statement holds for all r from 0 to $k-1$. Then we want to prove the corresponding inequality for $[a_0, a_2, \ldots, a_k, c]$ and $[a_0, a_2, \ldots, a_k, d]$. If we drop the first term of each, we have the opposite inequality to the one we are trying to prove between $[a_1, a_2, \ldots, a_k, c]$ and $[a_1, a_2, \ldots, a_k, d]$, by the inductive hypothesis (it falls under $r = k-1$). If we invert both sides of this inequality, we switch its direction to the type we are trying to prove, and then we add a_0 to both sides to prove the statement for $r = k$.

7. (Niven 7.3.1 & 7.3.2)

(a) Evaluate $[1, 1, 1, \ldots]$.

This infinite continued fraction converges to a number x that must satisfy the equation

$$x = 1 + \frac{1}{x},$$

by the recursion in the definition of the continued fraction. We can solve this by multiplying through by x.

$$x^2 - x - 1 = 0,$$

so the quadratic formula gives

$$x = \frac{1 \pm \sqrt{5}}{2}.$$

Since x is clearly greater than 1, because its first term in the continued fraction expansion is 1, we must have the positive square root. So

$$[1, 1, 1, \ldots] = \frac{1+\sqrt{5}}{2},$$

also known as ϕ, the golden ratio.

(b) Evaluate $[2, 1, 1, 1, \ldots]$.

$$[2, 1, 1, 1, \ldots] = 1 + [1, 1, 1, 1, \ldots] = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}.$$

(c) Evaluate $[2, 3, 1, 1, 1, \ldots]$.

$$[2, 3, 1, 1, 1, \ldots] = 2 + \frac{1}{2 + [1, 1, 1, \ldots]} = 2 + \frac{1}{2 + \frac{1+\sqrt{5}}{2}}$$
$$= 2 + \frac{2}{5+\sqrt{5}} = 2 + \frac{10 - 2\sqrt{5}}{20} = \frac{25 - \sqrt{5}}{10}.$$

8. (Niven 7.3.4) For $i \geq 1$, prove that

$$\frac{k_i}{k_{i-1}} = [a_i, a_{i-1}, \ldots, a_2, a_1].$$

Find and prove a similar formula for $\frac{h_i}{h_{i-1}}$ (*Hint: Use the Euclidean algorithm on k_i and* k_{i-1}).

We have a recursion to find the k_i's that is: $k_{-2} = 1$, $k_{-1} = 0$, and $k_i = a_i k_{i-1} + k_{i-2}$ for $k \geq 0$. So clearly $k_0 = 1$ and $k_1 = a_1$. This proves the above formula for $i = 1$. Now assume inductively that the formula holds for $i < n$. Then $k_n = a_n k_{n-1} + k_{n-2}$ implies

$$\frac{k_n}{k_n - 1} = a_n + \frac{k_{n-2}}{k_{n-1}} = a_n + \frac{1}{\frac{k_{n-1}}{k_{n-2}}} = a_n + \frac{1}{[a_{n-1}, a_{n-2}, \ldots, a_1]} = [a_n, a_{n-1}, \ldots, a_1].$$

For the h_i's the difference in the recursion only comes from the switching of the values of h_{-2} and h_{-1}, which leads to the base case being instead

$$\frac{h_0}{h_{-1}} = a_0.$$

The rest of the proof follows exactly, and you get that for all $i \geq 0$,

$$\frac{h_i}{h_{i-1}} = [a_i, a_{i-1}, \ldots, a_0].$$

9. Calculate the first three convergents for

(a) e^2 (b) 2π.

(a) $e^2 = [7, 2, 1, 1, \ldots]$. So the first three convergents are 7, $\frac{15}{2}$, and $\frac{22}{3}$.

(b) $2\pi = [6, 3, 1, 1, \ldots]$. So the first three convergents are 6, $\frac{19}{3}$, and $\frac{25}{4}$.

10. Calculate the infinite continued fraction expansions for

(a) $\sqrt{7}$ (b) $\frac{1+\sqrt{13}}{2}$.

(a)

$$\begin{aligned}
\sqrt{7} &= 2 + (\sqrt{7} - 2) \\
\frac{1}{\sqrt{7} - 2} &= \frac{\sqrt{7} + 2}{3} = 1 + \frac{\sqrt{7} - 1}{3} \\
\frac{3}{\sqrt{7} - 1} &= \frac{3(\sqrt{7} + 1)}{6} = 1 + \frac{\sqrt{7} - 1}{2} \\
\frac{2}{\sqrt{7} - 1} &= \frac{2(\sqrt{7} + 1)}{6} = 1 + \frac{\sqrt{7} - 2}{3} \\
\frac{3}{\sqrt{7} - 2} &= \frac{3(\sqrt{7} + 2)}{3} = 4 + (\sqrt{7} - 2)
\end{aligned}$$

We have repeated a remainder, so it is clear that the continued fraction expansion is $[2, \overline{1, 1, 1, 4}]$.

(b)

$$\frac{1+\sqrt{13}}{2} = 2 + \frac{\sqrt{13}-3}{2}$$
$$\frac{2}{\sqrt{13}-3} = \frac{2(\sqrt{13}+3)}{4} = 3 + \frac{\sqrt{13}-3}{2}$$

Immediately we get a repeated remainder! So the continued fraction expansion is simply $[2, \overline{3}]$.

(Bonus) Let n be a positive integer.

(a) Prove that $\sqrt{n^2+1} = [n, \overline{2n}]$.

(b) Prove that $\sqrt{n^2+2} = [n, \overline{n, 2n}]$.

(c) Prove that $\sqrt{n^2+2n} = [n, \overline{1, 2n}]$.