18.786 Problem Set 2 - Spring 2008

Due Thursday, Feb. 28 at 1:00

Adopt the notation $\zeta_n := e^{2\pi i/n}$ and $\tilde{\zeta}_n := \zeta_n + \zeta_n^{-1}$.

1. Show that $\mathbb{Z}[x]$ contains prime ideals that are not maximal.

2. Calculate the norm $N_{L/\mathbb{Q}}(\alpha)$ and trace $T_{L/\mathbb{Q}}(\alpha)$ for:

   (a) $L = \mathbb{Q}[\zeta_5], \alpha = \tilde{\zeta}_5$,

   (b) $L = \mathbb{Q}[\zeta_6], \alpha = \zeta_6$,

   (c) $L = \mathbb{Q}[\sqrt[3]{5}], \alpha = 2 + \sqrt[3]{5}$.

3. If $R = \mathbb{Z}[\sqrt{5}]$, find an example of a local ring that is not a discrete valuation ring.

4. Let $R = \mathbb{Z}[\sqrt{-5}]$, and define the fractional ideal $U = \frac{1}{2}\mathbb{Z} + \frac{1}{1+\sqrt{-5}}\mathbb{Z}$. Show that U^{-1} is not a principal ideal. (*Hint:* It is not necessary to completely determine U^{-1}; also keep in mind that the class number for this field is 2).

   (Optional) Find other examples of fractional ideals whose inverses are not principal.

5. Exercise 2 on page 17 of Janusz.

6. (a) Prove that $F[x]$ is a Dedekind domain for any field F.

   Hint: Use condition 2 of Theorem I.3.16 in Janusz. If $F = \mathbb{C}$ the connection should be apparent between the uniformizer of a local ring and the local variable $(x - a)$ used in the Laurent series expansion about $x = a$ for a complex function).

   (b) Let $P(x, y) \in \mathbb{C}[x, y]$ be an irreducible, nonsingular polynomial, so that P is coprime to its (partial) derivatives. Prove that $\mathbb{C}[x, y]/(P)$ is a Dedekind domain.

   Hint: Hilbert's Nullstellensatz implies that the ideals in a polynomial ring are associated to their zero sets. In particular, maximal ideals are associated to points, and for ideals in $\mathbb{C}[x, y]/(P)$, these must be points at which P vanishes. Here you will see the local ring unformizer as the uniformizing parameter along the zero locus of P.

7. In this problem, you will prove the van der Monde determinant formula in two ways. The statement is that

$$\det \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j).$$

   (a) Prove the formula directly, by inducting on n.

   (b) Prove the formula by using unique factorization in polynomial rings over $\mathbb{R}$ (or any field): show that both sides of the equation have the same roots, degree, and leading coefficient.

8. (a) Use the quadratic formula and the minimal polynomial to explicitly write down $\tilde{\zeta}_5$ as a radical expression.

(b) In the remainder of this problem you will prove that if p is an odd prime, then $\sqrt{\varepsilon p} \in \mathbb{Q}[\zeta_p]$, where $\varepsilon = 1$ if $p \equiv 1, 2 \pmod 4$ and $\varepsilon = -1$ otherwise.

i) Show that the field discriminant of the extension $\mathbb{Q}[\zeta_p]/\mathbb{Q}$ is $\pm p^{p-2}$.

ii) Use the van der Monde determinant to conclude the stated claim.

9. Find the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[x]/(x^3 - 2x + 5)$ (use the discriminant).

10. (Computational) Calculate the class group of $\mathbb{Q}[\sqrt{d}]$ for all square-free integers $|d| < 500$. Next, calculate the class group for the splitting fields $\mathbb{Q}[x]/(x^3 + d)$ for all cube-free $|d| < 500$. Describe any interesting patterns that you see, particularly regarding the prime factors of the class numbers. Turn in a printout of the commands you used (but not all of your output!).