18.786 Problem Set 5 - Spring 2008

Due Thursday, Apr. 3 at 1:00

1. Exercise 1 on page 73 of Janusz.

2. Let K be an abelian number field whose Galois group is $G_{K/\mathbb{Q}} \cong (\mathbb{Z}/2\mathbb{Z})^n$ for some $n \geq 0$, such that all primes $2 \neq p \in \mathbb{Z}$ are unramified. Show that K must be a subfield of $\mathbb{Q}(i, \sqrt{-2})$, so $n \leq 2$.

3. Use Minkowski's bound (i.e., pen and paper!) to find the class numbers of the following fields:

   (a) $\mathbb{Q}[\sqrt{-23}]$

   (b) $\mathbb{Q}[\sqrt{-47}]$

4. Let K be a cubic number field with $\Delta(K) < 0$. Use the Minkowski bound to find a value for M such that $\mathrm{cl}(K) = 1$ whenever $-M < \Delta(K) < 0$.

5. Characterize the ring of units (use the unit Theorem to find the rank, torsion subgroup, and fundamental unit when possible/meaningful) for the following number fields:

   (a) $\mathbb{Q}[\sqrt{-7}]$

   (b) $\mathbb{Q}[\sqrt{7}]$

   (c) $\mathbb{Q}[\zeta_5]$