18.786 Problem Set 6 - Spring 2008

Due Thursday, Apr. 10 at 1:00

1. Characterize the ring of units (use the unit Theorem to find the rank, torsion subgroup, and fundamental units when possible/meaningful) for the following number fields:

   (a) $\mathbb{Q}\left[\tilde{\zeta_7}\right]$

   (b) $\mathbb{Q}[\sqrt{5}, \sqrt{-2}]$

   (c) $\mathbb{Q}[\zeta_n]$ (find the rank and fundamental units as a function of n)

2. (a) For any integers m and M, let $\mathcal{O}^{\star}_{m,M}$ be the set of all algebraic integers α over $\mathbb{Z}$ such that the degree of α is at most m, and $|\sigma(\alpha)| < M$ for all conjugates of α. Prove that $\mathcal{O}^{\star}_{m,M}$ is finite.

   Hint: Any such α has a minimal polynomial – bound the coefficients.

   (b) Let $\mathcal{O}_{m,M}$ be the set of all algebraic integers α over $\mathbb{Z}$ such that the degree of α is at most m and $|\alpha| < M$. Is $\mathcal{O}_{m,M}$ finite?

3. Let K be either of the number fields $\mathbb{Q}(\sqrt{\pm 2}, \sqrt{-5})$.

   (a) Prove that $\mathcal{O}_K^{\times}$ has rank 1.

   (b) Use ramification in cyclotomic fields to prove that the torsion subgroup $T(K)$ is $\{\pm 1\}$.

4. Let K be a number field, and let S be a finite set of nonzero primes in $\mathcal{O}_K$, with localization $\mathcal{O}_{K,S}$ (localize with respect to ideals whose only prime factors are elements of S).

   (a) Prove that the torsion subgroups of $\mathcal{O}_{K,S}^{\times}$ and $\mathcal{O}_K^{\times}$ are isomorphic.

   (b) Prove that $\mathcal{O}_{K,S}^{\times}/\mathcal{O}_K^{\times}$ is a free $\mathbb{Z}$-module of rank $\#S$.

   Hint: Use the finiteness of the class group – principal ideals are easier to factor.

5. A *Salem number* is a real algebraic integer $\alpha > 1$ such that $|\sigma(\alpha)| \leq 1$ for all of its conjugates, with equality achieved at least once. Find an example.