18.786 Problem Set 8 - Spring 2008

Due Thursday, May 1 at 1:00

1. Exercise 3 on page 118 of Janusz.
2. Exercise 5 on page 119 of Janusz.
3. Exercise 7 on page 119 of Janusz.
4. (a) Show that if $|\bullet|_1, \ldots, |\bullet|_n$ are inequivalent absolute values on a field K, then there is an element $a \in K$ such that $|a|_1 > 1$ and $|a|_2, \ldots, |a|_n < 1$.

   (b) For any $1 \leq r \leq n$, show that a can be chosen so that $|a|_1, \ldots, |a|_r > 1$ and $|a|_{r+1}, \ldots, |a|_n < 1$.

   (c) If $K = \mathbb{Q}$, reinterpret part (a) as a statement about rational numbers (remember to include the norm $|\bullet|_\infty$ in your discussion). Illustrate with an example.
5. Determine whether or not each of the integers $8, 9, \ldots, 13$ are square in $\mathbb{Z}_7$. For each of the squares $m \in [8, 13]$, find the 7-adic expansion of $\sqrt{m}$.
6. Define the n-adic norm for composite n analogously to the p-adic ones, so $|\frac{a}{b} \cdot n^r|_n := n^{-r}$, where $n \nmid a, b$. Let $\mathbb{Z}_n$ be the completion of $\mathbb{Z}$ with respect to this norm.

   (a) Show that $\mathbb{Z}_{10}$ is not an integral domain (i.e., it has zero divisors).

   (b) Can you find a composite n such that $\mathbb{Z}_n$ still is an integral domain?
7. Use Hensel's lemma to prove that -9 has a cube root in $\mathbb{Q}_{10}$.