18.786 Problem Set 9 - Spring 2008

Due Thursday, May 8 at 1:00

1. Exercise 2 on page 118 of Janusz.

2. Let $K = \mathbb{Q}(\sqrt[3]{5})$ and find all of the extensions to K to the following valuations on $\mathbb{Q}$:

   (a) $|\bullet|_3$

   (b) $|\bullet|_5$

   (c) $|\bullet|_\infty$ (i.e., absolute value)

3. Determine the 7 non-isomorphic quadratic extensions of $\mathbb{Q}_2$.

4. If K is a number field, recall that the zeta-function for K is

$$\zeta_K(s) = \sum_{\mathfrak{p}\text{ prime}} \mathrm{Nm}(\mathfrak{p})^{-s}.$$

   (a) Prove that for $\Re(s) > 1$,

$$\zeta_{\mathbb{Q}}(s) = \frac{1}{1 - 2 \cdot 2^{-s}} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s}.$$

   Remark: This shows that ζ has a simple pole at $s = 1$.

   (Optional) Find a more general version of the above identity.

   (b) Calculate the probability that an element of $\mathcal{O}_K$ is k-th power free (e.g. squarefree) in terms of ζ_K. You may assume that $\mathcal{O}_K$ is a PID.

5. (Computational) Consider at least two different number fields and tabulate the factorizations of all primes $p < 1000$. In particular, calculate the ratio of primes that split completely and compare with the $1/[K : \mathbb{Q}]$ density predicted by the Chebotarev density theorem.

(Optional) State and prove an analog of the product formula for the function field case $K = F(x)$ (where F is a field).