MATH 18.02A - Mar. 5 Recitation

1. a) Consider the surface defined by an equation $z = f(x, y)$, where f is a smooth function. Parameterize the surface by

$$x = x, y = y, z = f(x, y)$$

and calculate the area element dA. $\left(Ans: \sqrt{1 + f_x^2 + f_y^2}\right)$

Hint: Recall that $d\mathbf{S} = T_x \times T_y$, *and that* $dA = |d\mathbf{S}|$.

b) Suppose that a thin metal shell in the first quadrant is described by $z = 4 - x - y^2$. If the shell has charge density $\delta(x, y, z) = y$, set up an integral that calculates the total charge (you do not need to evaluate). $\left(Ans: \int_0^4 \int_0^{\sqrt{4-x}} y\sqrt{2 + 4y^2}\, dydx = \frac{484\sqrt{2}}{15}\right)$

2. Determine which of the following fields are conservative (you may not need to calculate the entire curl if the answer is no!):

a) $\mathbf{F} = (x + 2y)\,\hat{\mathbf{\imath}} + (3x + 2z)\,\hat{\mathbf{\jmath}} + (-4x)\,\hat{\mathbf{k}}$. (*Ans: No*)

b) $\mathbf{F} = 2x\,\hat{\mathbf{\imath}} + (2x + 2y)\,\hat{\mathbf{\jmath}} + \left(14\cos(y + x)e^{\sqrt{1+x^3-3z^2}} - \ln(4x - 3y)\right)\hat{\mathbf{k}}$. (*Ans: No*)

c) $\mathbf{F} = z\cos(x + y)\,\hat{\mathbf{\imath}} + (z\cos(x + y) - e^z)\,\hat{\mathbf{\jmath}} + (\sin(x + y) - ye^z)\,\hat{\mathbf{k}}$. (*Ans: Yes*)

3. a) Let $\mathbf{F} = (3y - x)\,\hat{\mathbf{\imath}} + (x - 3z)\,\hat{\mathbf{\jmath}} + (x - y)\,\hat{\mathbf{k}}$, and let $\mathbf{c}$ be the counterclockwise path around the triangular boundary of $2x + y + z = 2$ in the first quadrant. Use Stokes' Theorem to evaluate $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$. (*Ans:* 8)

b) Let S be the top-half of the ellipsoid $x^2 + y^2 + \frac{z^2}{4} = 1$, oriented with the normals pointing away from the origin. If

$$\mathbf{F} = (x^3 + 2y)\,\hat{\mathbf{\imath}} + y^4e^y\,\hat{\mathbf{\jmath}} + z^2\sin(x^2 + y)\,\hat{\mathbf{k}},$$

evaluate $\iint_S \operatorname{curl}\mathbf{F} \cdot \hat{\mathbf{n}}\, dA$ by replacing S with a simpler surface that shares the same boundary. (*Ans:* -2π)

Remark. Part b) illustrates a significant difference between Stokes' Theorem and Green's Theorem: the integral of the curl over a complicated surface may be replaced by a computationally simpler surface so long as they share the same boundary (there wasn't any "space" to do this in two dimensions). This freedom of choice is also useful when using Stokes' Theorem to evaluate line integrals, as any surface with the right boundary may be used.