18.02A Exam 1 Review Solutions - Spring 2007

1. (*Changes of variables*)

a) The Jacobian is $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right|^{-1} = \frac{1}{4}$. The region is the triangle bounded by $(0,0), (1,0)$, and $(0,1)$, so the integral is

$$\int_0^1 \int_0^{1-x} (x+y)^3 \sin(x-y)\, dydx$$
$$= \int_0^1 \int_{-u}^{u} (u)^3 \sin(v) \cdot \frac{1}{2}\, dvdu = \boxed{0} \quad (\textit{odd function}).$$

b) The Jacobian is $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \left|\begin{smallmatrix} 2 & 1 \\ -2 & 1 \end{smallmatrix}\right|^{-1} = \frac{1}{4}$. The four boundary lines translate to $u = 0, u = 2, v = 0$, and $v = 2$, so the integral is

$$\iint_R (y^2 - 4x^2)e^{8x^2+2y^2}\, dA$$
$$= \int_0^2 \int_0^2 uve^{u^2+v^2} \cdot \frac{1}{4}\, dudv = \boxed{\frac{(e^4-1)^2}{16}}.$$

c) Now the Jacobian is

$$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right|^{-1} = \begin{vmatrix} 1 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & z & y \end{vmatrix}^{-1} = \frac{1}{y}.$$

The (x,y,z) first octant is also bounded by $u, v, w \geq 0$, and the other two curves become $u + v^2 = 1$ and $w = 2$. Finally, $x = 2v$ and $y = \frac{u-2v}{2}$, so the integral is

$$\iiint_R \frac{1}{xy}\, dV = \int_0^2 \int_0^1 \int_0^{1-v^2} \frac{1}{xy} \cdot \frac{-1}{y}\, dudvdw$$
$$= \boxed{\int_0^2 \int_0^1 \int_0^{1-v^2} \frac{2}{v(u-2v)^2}\, dudvdw}.$$

2. (*Triple integrals*)

a) The bounding plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Let $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$. Then

$$M_x = \iiint_R x\, dV = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} au \cdot abc\, dwdvdu$$
$$= a^2bc \int_0^1 \int_0^{1-u} u(1-u-v)\, dvdu = a^2bc \int_0^1 \frac{u}{2}(1-u)^2\, du = \frac{a^2bc}{24}.$$

By the prism formula, the volume of the region is $\frac{abc}{6}$, so $\bar{x} = \frac{a}{4}$. Symmetry implies that the center of mass is $\boxed{\left(\frac{a}{4}, \frac{b}{4}, \frac{c}{4}\right)}$.

b) Using cylindrical coordinates,

$$I_z = \int_0^{2\pi}\int_0^a\int_0^{b(1-\frac{r}{a})} r^3 \, dzdrd\theta$$
$$= b\int_0^{2\pi}\int_0^a r^3 - \frac{r^4}{a}\, drd\theta = b\int_0^{2\pi} \frac{a^4}{4} - \frac{a^4}{5}\, d\theta = \boxed{\frac{\pi b a^4}{10}}.$$

c) The mass of the first sphere is twice that of the top hemisphere. In spherical coordinates, this is

$$2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a \gamma\rho\cos\phi\cdot\rho^2\sin\phi\, d\rho d\phi d\theta = \frac{\gamma a^4}{2}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos\phi\sin\phi\, d\phi d\theta$$
$$= \frac{\gamma a^4}{2}\int_0^{2\pi}\frac{1}{2}\, d\theta = \boxed{\frac{\pi a^4}{2}}.$$

The second sphere has mass density r^2, so the total mass is

$$\int_0^{2\pi}\int_0^{\pi}\int_0^a \gamma\rho^2\sin^2\phi\cdot\rho^2\sin\phi\, d\rho d\phi d\theta = \frac{\gamma a^5}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\phi\, d\phi d\theta$$
$$= \frac{\gamma a^5}{5}\int_0^{2\pi}\int_0^{\pi}(1-\cos^2\phi)\sin\phi\, d\phi d\theta == \frac{\gamma a^5}{5}\int_0^{2\pi}\frac{4}{3}\, d\theta = \boxed{\frac{8\pi a^5}{15}}.$$

3. (*Vector fields*)

a) See Figure 1

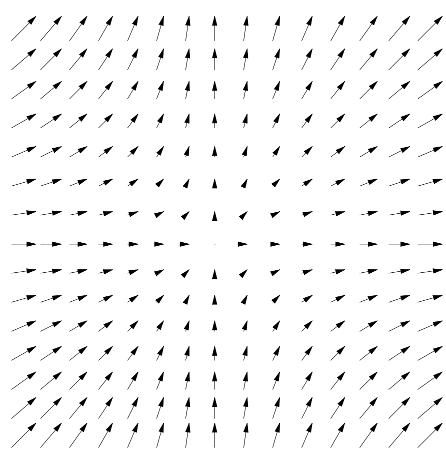

Figure 1: $\mathbf{F} = |x|\,\hat{\mathbf{\imath}} + |y|\,\hat{\mathbf{\jmath}}$

b) See Figure 2.

c) See Figure 3

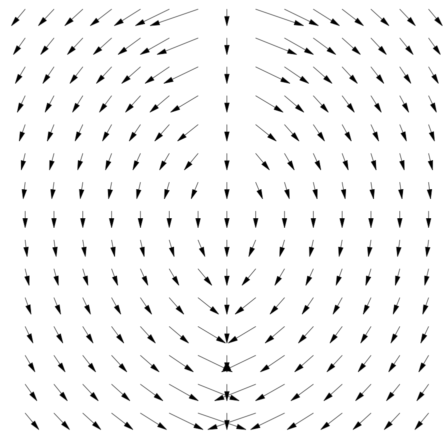

Figure 2: $\mathbf{F} = \frac{y}{x}\,\hat{\imath} - \hat{\jmath}$

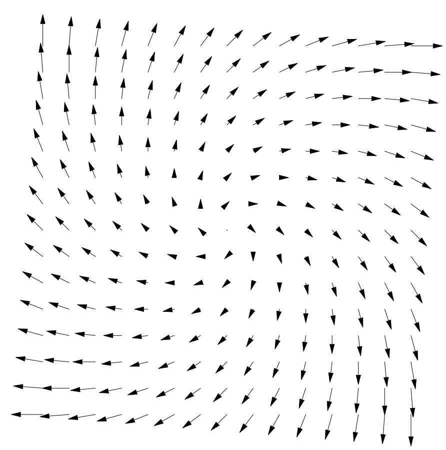

Figure 3: $\mathbf{F} = (x+y)\,\hat{\imath} + (y-x)\,\hat{\jmath}$

d) $\sqrt{x^2+y^2}\,\hat{\jmath}$.

e) $\frac{y\,\hat{\imath}-x\,\hat{\jmath}}{2}$ or $\frac{-y\,\hat{\imath}+x\,\hat{\jmath}}{2}$.

4. (*Line integrals*)

a) Using the obvious parameterizations for the coordinate axes, and (t,t) for the diagonal,

$$\begin{aligned}\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 -x\,dx + \int_0^1 -t^2\,dt + \int_1^0 (t-t) - t^2\,dt \\ &= \int_0^1 -x\,dx = \boxed{\frac{-1}{2}}.\end{aligned}$$

b) In general, for a constant field $\mathbf{F} = \mathbf{a}$, the line integral along a straight line from P_0 to P_1 is the projection of the path in the direction of $\mathbf{a}$ scaled by the magnitude of $\mathbf{a}$. In other words, if $\mathbf{a}$ is the vector from P_0 to P_1, then the line integral is just $\mathbf{a}\cdot\mathbf{v}$.

Here, the first path follows the vector $(2,0)$, so the integral is

$$\int_{(1,0)}^{(3,0)} \mathbf{F}\cdot d\mathbf{r} = (1,-1)\cdot(2,0) = \boxed{2}.$$

The second path has integral

$$\int_{(-1,1)}^{(1,-1)} \mathbf{F}\cdot d\mathbf{r} = (1,-1)\cdot(2,-2) = \boxed{4}.$$

c) For the first path,

$$\int_{(1,0)}^{(3,0)} \mathbf{F}\cdot d\mathbf{r} = \int_1^3 1\,dx = \boxed{2}.$$

Parameterize the second path as $(t,-t)$ to get

$$\int_{(-1,1)}^{(1,-1)} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} 1-(-1)\,dt = \boxed{4}.$$

d) This is a straightforward evaluation:

$$\int_{\mathbf{c}} (x+\sin z)\,dx + (4-x^2)\,dy + 3y\,dz$$
$$= \int_0^{2\pi} \left[(\sin t+\sin t)\cos t + (4-\sin^2 t)\sin t - 3\cos t\right]\,dt = \boxed{0} \qquad (\textit{by periodicity}).$$

5. (*Conservative/Path-independent/Gradient fields*)

a) i) In the clockwise direction, starting from $(2,1)$,

$$\oint_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{r} = \int_2^{-2} 1\,dx + \int_1^{-1} -4\,dy + \int_{-2}^{2} 1\,dx + \int_{-1}^{1} 4\,dy = \boxed{0}.$$

ii) $\boxed{\textbf{No,}}$ since curl $\mathbf{F} = -2x \pm 1$ (or is undefined) depending on the sign of y.

b) Must have curl $\mathbf{F} = \frac{2}{x^2} - (\frac{-2}{x^2}) = 0$, but this is impossible; there are **no values of** a.

c) Evaluate at the endpoints, Let $\mathbf{F} = \vec{\nabla}(x^2 + \tan^{-1}(xy))$, and evaluate

$$\int_{(0,1)}^{(\sqrt{3},1)} \mathbf{F}\cdot d\mathbf{r} = (3+\tan^{-1}\sqrt{3}) - \tan^{-1} 0 = \boxed{3+\frac{\pi}{3}}.$$

d) The integral is path independent, since

$$\text{curl} = \frac{\pi}{2}\left(\cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right) - \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)\right) = 0.$$

Thus the path may be replaced by straight lines following the coordinate axes, so

$$\int_{\mathbf{c}} \sin\left(\frac{\pi y}{2}\right)\cos\left(\frac{\pi x}{2}\right)\,dx + \sin\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi y}{2}\right)\,dy$$
$$= \int_{-1}^{0} 0\,dy + \int_0^1 0\,dx = \boxed{0}.$$

6. (*Potential functions*)

a) A potential function is $f(x, y, z) = \boxed{3z\cos x - 2(x+y)^2 + c}$.

b) Need curl $\mathbf{F} = -2y - (-2y) = 0$, which is true for all values of a. Both the algebraic and integration methods require the use of integration by parts, and the potential functions are

$$f(x,y) = \boxed{\begin{cases} \frac{\cos(ax)+x\sin(ax)}{a} - xy^2 + c & \text{if } a \neq 0, \\ \frac{x^2}{2} - xy^2 + c & \text{if } a = 0. \end{cases}}$$

c) After verifying that the curl is zero,

$$f(x,y) = \boxed{\begin{cases} \frac{(x^2+y^2)^{b+1}}{2(b+1)} + c & \text{if } b \neq -1, \\ \frac{1}{2}\ln(\sqrt{x^2+y^2}) + c & \text{if } b = -1. \end{cases}}$$