18.02A Exam 1 Solutions - Spring 2007

Problem 1. a) Recall that the gradient of $f(x, y)$ is always perpendicular to any level curve $f(x, y) = C$. Therefore, the line integral of $\vec{\nabla} f$ along such a level curve is $\boxed{\mathbf{0}}$.

b) The missing values are for the Jacobian $\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}^{-1}$. In this case, the integral expression should be filled in as

$$\iiint_{R_{xyz}} f(x, y, z)dV = \iiint_{R_{uvw}} g(u, v, w) \begin{vmatrix} \boldsymbol{2x} & \mathbf{0} & \mathbf{1} \\ \boldsymbol{\sin y} & \boldsymbol{x \cos y} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} \end{vmatrix}^{-1} dudvdw.$$

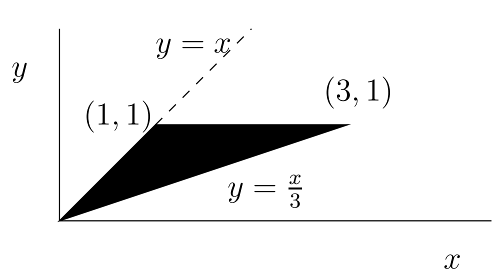
c) Any constant field is conservative, so a straight line path $\mathbf{c}(t) = (x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0))$ may be used. The line integral is then

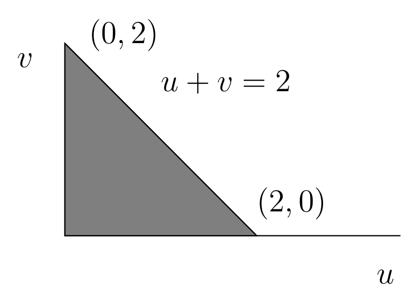
$$\begin{aligned} \int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)\, dt \\ &= \boxed{\boldsymbol{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}}. \end{aligned}$$

d) The radially inward unit vector field is $\mathbf{F} = \frac{-x\,\hat{\imath} - y\,\hat{\jmath}}{\sqrt{x^2+y^2}}$, and a right-angled clockwise rotation sends a vector (a, b) to $(b, -a)$. Thus

$$\mathbf{F} = \boxed{\boldsymbol{\frac{-y\,\hat{\imath} + x\,\hat{\jmath}}{\sqrt{x^2 + y^2}}}}.$$

Problem 2. The triangle is bounded by the lines $y = x$, $y = \frac{x}{3}$, and $y = 1$. In (u, v) coordinates these lines become $v = 0, u = 0$, and $u + v = 2$, respectively. Furthermore, the (x, y) points $(0, 0), (1, 1)$, and $(3, 1)$ are converted to the (u, v) points $(0, 0), (2, 0)$, and $(0, 2)$. The regions are displayed in the following figures:





Next, calculate the Jacobian

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}^{-1} = \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix}^{-1} = \frac{1}{2}.$$

Therefore

$$\iint_R 2(x-3y)^2(x-y)\,dA = \int_0^2 \int_0^{2-u} 2u^2v \cdot \frac{1}{2}\,dvdu$$
$$= \int_0^2 \frac{u^2(2-u)^2}{2}\,du = \int_0^2 2u^2 - 2u^3 + \frac{u^4}{2}\,du = \frac{16}{3} - \frac{32}{4} + \frac{16}{5} = \boxed{\mathbf{\frac{8}{15}}}.$$

Problem 3. The region can be pictured as an upside-down bowl. Using cylindrical coordinates, the mass is

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r \cdot r\,dzdrd\theta = \int_0^{2\pi}\int_0^2 r^2(4-r^2)\,drd\theta$$
$$= \int_0^{2\pi} \frac{4\cdot 2^3}{3} - \frac{2^5}{5}\,d\theta = \boxed{\mathbf{\frac{128\pi}{15}}}.$$

Problem 4. a) $\mathbf{F}$ is conservative if

$$0 = \operatorname{curl}\mathbf{F} = 6ax^5y - 3e^y - (12x^5y - 3e^y) = 6x^5y(a-2),$$

so $\boxed{\boldsymbol{a = 2}}$.

b) Using the algebraic method, if $\mathbf{F} = \vec{\nabla} f(x,y)$, then

$$f_x = 6x^5y^2 + \cos x - 3e^y.$$

Thus $f(x,y) = x^6y^2 + \sin x - 3xe^y + g(y)$ for some function g. Matching this with the other component of $\mathbf{F}$ implies that

$$f_y = 2x^6y - 3xe^y + g_y = 2x^6y - 3xe^y.$$

Therefore g is a constant function, and

$$f(x,y) = \boxed{\boldsymbol{x^6y^2 + \sin x - 3xe^y + C}}.$$

c) The work done by $\mathbf{F}$ on a path is simply the line integral $\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{r}$, and the Fundamental Theorem of Calculus for line integrals states that this integral is simply f evaluated at the endpoints. These points are $(0,1)$ and $(-3,2)$ in this case, so the work is

$$\int_{(0,1)}^{(-3,2)} \mathbf{F}\cdot d\mathbf{r} = \int_{(0,1)}^{(-3,2)} \vec{\nabla} f \cdot d\mathbf{r} = f(-3,2) - f(0,1) = \boxed{\mathbf{2916 - \sin 3 + 9}\boldsymbol{e}^{\mathbf{2}}}.$$

Problem 5. a) Check both fields by calculating the curl:

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= 4x,\\ \operatorname{curl} \mathbf{G} &= 3x^2 - 3x^2 = 0,\end{aligned}$$

so $\boxed{\mathbf{F}}$ is not conservative.

To verify this, select two paths from $(0, 0)$ to $(1, 1)$. The first path $\mathbf{c}_1$ travels along the x-axis to $(1, 0)$, and then rises to the final point; the path $\mathbf{c}_2$ travels up the y-axis to $(0, 1)$ before proceeding to the end. The line integrals are

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dx + \int_0^1 2\, dy = 2,$$

and

$$\int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dy + \int_0^1 0\, dx = 0,$$

which are unequal.

b) Since $\mathbf{F}$ is conservative, $\operatorname{curl} \mathbf{F} = 0 = g'(x)$. Therefore, $g(x)$ must be a $\boxed{\textbf{constant}}$ function. There is $\boxed{\textbf{no restriction}}$ on $h(y)$.