18.02A Practice Questions – Exam 7

The actual exam will be about 1/3 first two weeks, 2/3 last two weeks. Use the last two problem sets to supplement the questions below on the last two weeks. (The practice questions below are evenly divided in terms of numbers, but the last five are more involved and would get more points. You should also review exam 6 and practice exam 6, for more problems using Green's theorem in both the tangential and normal form.)

Problem 1.

a) In the xy-plane, let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$. Give in terms of P and Q the line integral representing the flux of $\mathbf{F}$ across a simple closed curve C, with outward-pointing normal.

b) Let $\mathbf{F} = ax\mathbf{i} + by\mathbf{j}$. How should the constants a and b be related if the flux of $\mathbf{F}$ over any simple closed curve C is equal to the area inside C?

Problem 2.

A solid hemisphere of radius 1 has its lower flat base on the xy-plane and center at the origin. Its density function is $\delta = z$. Find the force of gravitational attraction it exerts on a unit point mass at the origin.

Problem 3.

Evaluate $\int_C (y-x)dx + (y-z)dz$ over the line segment C from $P : (1,1,1)$ to $Q : (2,4,8)$.

Problem 4.

Consider a solid sphere of radius a with center at the origin; let H be its solid upper hemisphere (i.e., the part above the xy-plane). Set up a triple integral in spherical coordinates which gives the average distance of a point in H from the xy-plane.

(Give integrand, limits, and the constant factor in front, but *do not evaluate.*)

Problem 5.

Let C be a solid right circular cone having base radius 1 and vertex angle $60°$. Set up an integral in cylindrical coordinates which represents the moment of inertia of C about its central axis; assume the density $\delta = 1$.

(Place the cone so its axis is the z-axis and its vertex is at the origin; supply integrand and limits, but *do not evaluate.*)

Problem 6.

a) Let $\mathbf{F} = ay^2\mathbf{i} + 2y(x+z)\mathbf{j} + (by^2 + z^2)\mathbf{k}$. For what values of the constants a and b will $\mathbf{F}$ be conservative? Show work.

b) Using these values, find a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.

c) Using these values, give the equation of a surface S having the property: $\int_P^Q \mathbf{F} \cdot d\mathbf{r} = 0$ for any two points P and Q on the surface S.

Problem 7.

Let S be the surface formed by the part of the graph of the paraboloid $z = x^2 + y^2$ lying below the plane $z = 1$, and let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + (1 - 2z)\mathbf{k}$.

Calculate the flux of $\mathbf{F}$ across S, taking the outward direction (i.e., the one pointing away from the z-axis) as the one for which the flux is positive. Do this two ways:

a) by a method which calculates $\iint_S \mathbf{F} \cdot d\mathbf{S}$ directly;

b) by using the divergence theorem.

Problem 8.

Let S be the infinite circular cylindrical surface given by the equation $x^2 + y^2 = 1$ having the whole z-axis as its central axis, and let $\mathbf{F} = (zx - y)\mathbf{i} + zy\mathbf{j} + z\mathbf{k}$.

a) Calculate $\nabla \times \mathbf{F}$ (i.e., curl $\mathbf{F}$).

b) Deduce that $\iint_R \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = 0$ for any finite portion R of the surface S.

c) Let C be any closed curve on S going once around S (and oriented as in the picture). Show by using the result of part (b) and Stokes' theorem that $\oint_C \mathbf{F} \cdot d\mathbf{r}$ always has a constant value independent of C, and determine this value.

Problem 9.

Let $\phi(x,y,z)$ be a function with continuous second partial derivatives.
Prove that $\nabla \times \nabla \phi = 0$

Problem 10.

An xz-cylinder in 3-space is a surface given by an equation $f(x,z) = 0$ in x and z alone; its section by any plane $y = c$ perpendicular to the y-axis is always the same xz-curve. (See picture.)

Show that if $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$, then $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any simple closed curve C lying on an xz-cylinder. (Use Stokes' theorem.)