18.02A Exam 2 Review - Spring 2007

There are several practice problems from each topic that may appear on the exam; you can expect approximately one problem from each of the numbered groups.

1. (*Triple integrals*)

a) Let R be the region satisfying $x^2 + y^2 + z^2 \leq 4$ and $-x \leq y \leq x$. Sketch the solid and find its moment of inertia about the z-axis.

b) Consider a cone of height b whose base is a circle of radius a in the x-y plane, centered at the origin. Find its total mass if the density is $\delta = |x|$.

c) Use a triple integral to find the volume of the region bounded by $z - x + y = 3, z = 0, x = 2$ and $y = 3$.

2. (*Line integrals*)

a) Let $\mathbf{c}$ be the path that travels in a straight line from $(0, 0)$ to $(2, 0)$, and then from $(2, 0)$ to $(0, 1)$. Calculate $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = (x^2 + 2xy)\,\hat{\mathbf{i}} - y^2\,\hat{\mathbf{j}}$.

b) Evaluate

$$\int_{\mathbf{c}} (y + 2x)\, dx + (1 - x^2)\, dy + 2z\, dz$$

along the path $\mathbf{c} = (t, 1 + t, 2 \sin \pi t)$ from $0 \leq t \leq 2$.

3. (*Conservative fields; Potential functions*)

a) Find the value of a such that

$$\mathbf{F} = (a + 3y \cos x + e^{x+y^2})\,\hat{\mathbf{i}} + (3 \sin x + aye^{x+y^2})\,\hat{\mathbf{j}}$$

is conservative. Using this value of a, find a potential function and evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$ for the path $\mathbf{c} = (3e^{t^2-1}, 2 - t)$ with $\quad -1 \leq t \leq 1$.

b) Determine which one of the following fields is conservative, and find its potential function:

$$\mathbf{F}_1 = ((yz + 1)e^{xyz})\,\hat{\mathbf{i}} + (xze^{xyz+x})\,\hat{\mathbf{j}} + (xye^{xyz+x})\,\hat{\mathbf{k}},$$
$$\mathbf{F}_2 = 3y\,\hat{\mathbf{i}} + (3x - \cos(y - z))\,\hat{\mathbf{j}} + (\cos(y - z) + 2z)\,\hat{\mathbf{k}}$$

4. (*Green's Theorem*)

a) Let $\mathbf{c}$ be the clockwise, triangular path with vertices $(0, 1), (2, 1)$ and $(1, 0)$. Calculate $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$ for the field $\mathbf{F} = 3y\,\hat{\mathbf{i}} + (2x + y)\,\hat{\mathbf{j}}$.

b) Let $\mathbf{c}$ be the counterclockwise path from $(2, 0)$ to $(-2, 0)$ along the circle of radius 2. Calculate the flux $\int_{\mathbf{c}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dr$ for the field $\mathbf{F} = x(1 - \cos y)\,\hat{\mathbf{i}} + (1 + \sin y)\,\hat{\mathbf{j}}$.

c) If $a, b \geq 0$, let $\mathbf{c}$ be the counterclockwise rectangular path with corners $(0, 0), (0, b), (-a, b)$ and $(-a, 0)$. For the field $\mathbf{F} = 2xy\,\hat{\mathbf{i}} + (x^2 + xy - y^2)\,\hat{\mathbf{j}}$, calculate both the work done along the path ($\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$) and the flux across the path ($\oint_{\mathbf{c}} \mathbf{F} \cdot \hat{\mathbf{n}}\, dr$).

5. (*Surface integrals*)

a) Let S_1 be the portion of the upper hemisphere ($z \geq 0$) with azimuthal angle $0 \leq \phi \leq \frac{\pi}{3}$, and let S_2 be the remaining part. Calculate the percentage of the total surface area contained on each of S_1 and S_2.

b) Find the surface area of a circular cone with base radius a and height b (i.e., the cone shrinks to a point at height b).

c) Write down an integral to calculate the surface area of the graph $z = 4 - x - y^2$ in the range $0 \leq x \leq 2, -1 \leq y \leq 1$.

d) Calculate the flux of $\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + (y - z^2)\,\hat{\mathbf{k}}$ through the cylindrical shell of radius 1 and height $0 \leq z \leq 2$, oriented with normals pointing away from the z-axis.

6. (*Stokes' Theorem*)

a) Let $\mathbf{F} = \sin(e^x)\,\hat{\imath} + z\,\hat{\jmath} + y^2\,\hat{\mathbf{k}}$, and $\mathbf{c} = (0, \cos t, \sin t)$ for $0 \leq t \leq 2\pi$. Calculate $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$.

b) Let $\mathbf{c}$ be the "circle tilted by angle α" with parameterization

$$x = \sin t, \quad y = -\cos t, \quad z = \tan\alpha(1 - \cos t), \qquad 0 \leq t \leq 2\pi.$$

Sketch the curve and explain why the curve can be described as a "tilted circle." Let $\mathbf{F} = (z - y)\,\hat{\imath} + y\,\hat{\jmath} + (x + z)\,\hat{\mathbf{k}}$, and calculate $\oint_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r}$.

c) If $\mathbf{c}$ is the triangle through $(1, 0, 0), (0, 1, 0)$ and $(0, 0, 1)$ in that order, find

$$\oint_{\mathbf{c}} xy\,dx + (x^2 + y^2)\,dy + (x + y + z)\,dz.$$

7. (*Divergence Theorem*)

a) Let S be the upper hemisphere of radius 2. Determine the flux of $\mathbf{F} = (x - 2xy)\,\hat{\imath} + y^2\,\hat{\jmath} + (y + 1)\,\hat{\mathbf{k}}$ through S.

b) Let S be the open rectangular box (i.e., 5 sides and an open top) with opposite corners at $(0, 0, 0)$ and (a, b, c), oriented with its normals pointing towards the exterior. Find the flux of $\mathbf{F} = (-x + 3y^2e^z)\,\hat{\imath} + (yx - z^2)\,\hat{\jmath} + z\,\hat{\mathbf{k}}$ through S.

c) If $\mathbf{F} = (-xy^2 - z)\,\hat{\imath} + 9y\,\hat{\jmath} + (-zx^2 - z^3)\,\hat{\mathbf{k}}$, determine the closed surface S that has maximum flux $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dA$.

8. (*Gravitation; Electromagnetism*)

a) The upper half of a thin hemispherical shell ($z \geq 0$) of radius a has charge density $\delta = z^2$. Find the total gravitational force exerted on a point of mass m placed at the origin.

b) Suppose that a circular wire loop of radius 2 lies in the (x, y)-plane centered around the origin. Calculate the electromagnetic force generated by the magnetic field $\mathbf{B} = 2x\sin t\,\hat{\imath} + (y - z^2\cos t)\,\hat{\jmath} + (t + x)\,\hat{\mathbf{k}}$ by using Faraday's Law:

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{r} = \frac{-1}{c}\frac{d}{dt}\iint_S \mathbf{B} \cdot \hat{\mathbf{n}}\,dA.$$