18.02A Problem Set 5 - Spring 2007

Due Friday, Feb. 16 at 2:00

Part I (10 points)

Lecture 40. (*Tues., Feb. 6*) Changes of variables in double integrals.
Read: Notes CV *Work:* 3D-1, 2, 9

Lecture 41. (*Thurs., Feb. 8*) Triple integrals - rectangular and cylindrical coordinates.
Read: Notes I.3; Simmons 20.5 *Work:* 5A-2acd, 4

Lecture 42. (*Frid., Feb. 9*) Changes of variables: Spherical coordinates; Gravitation.
Read: Notes G; Simmons 20.6, 20.7 *Work:* 5B-1a, 2

Lecture 43. (*Tues., Feb. 13*) Vector fields; Line integrals (two and three dimensions).
Read: Notes V1, V8, V11; Simmons 21.1 *Work:* 4A-3bc; 4B-1acd, 3; 6A-3

Lecture 44. (*Thurs., Feb. 15*) Conservative vector fields; Path independence; Gradient fields.
Read: Simmons 21.2 *Work:* 4C-1, 2, 3

Lecture 45. (*Fri., Feb. 16*) Exact differentials; Potential functions.
Read: Notes V2

Part II (15 points)

Try each problem alone for 15 minutes before collaborating, and write up solutions independently. The problems are given in order according to the lecture schedule above.

Problem 1. (*3 pts*) Make a substitution to evaluate the integral $\iint_R x\, dA$, where R is the ellipse bounded by

$$\left(\frac{x-2y}{2}\right)^2 + (2x+y)^2 = 1.$$

Problem 2. (*3 pts: 1+2*) A circus tent is constructed to have the shape bounded by the curves $x^2+y^2=R^2$, $z=0$, and $z=100e^{-x^2-y^2}$.
a) Find the total volume inside the tent.
b) If a fire breaks out and the tent is filled with smoke with density function $\delta = \alpha z$, calculate the average smoke density (= total smoke / total volume).

Problem 3. (*4 pts: 1+3*) a) Use spherical coordinates to find the volume of the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1.$$

b) Find the center of mass if the ellipsoid is shifted up a distance c along the z-axis (note that it will be tangent to the origin), and then cut in half by the plane $z=c$. *Hint: It will be much easier to find the average z value for a half ellipsoid centered at the origin, and then "flip" it at the final step.*

Problem 4. (*3 pts: 1+1+1*) Evaluate the line integral $\int_P \mathbf{F} \cdot d\mathbf{r}$ for the paths from the origin to $A = (1, 0)$ given below, where

$$\mathbf{F} = \frac{2x}{(1+y)^2}\,\hat{\mathbf{i}} - \frac{2x^2}{(1+y)^3}\,\hat{\mathbf{j}}.$$

a) P_1 is a straight line along the x-axis.
b) P_2 travels along $y = x$ until reaching the point $(1, 1)$, and then drops vertically to A.
c) P_3 travels in straight line segments to $(0, -1)$, then $(1, -1)$, and finally to A.

Problem 5. (*2 pts*) It is a fact that the vector field $\mathbf{F} = x\,\hat{\mathbf{i}}$ is conservative. Show that, however, $\mathbf{G} = x\,\hat{\mathbf{j}}$ is not (by finding two paths between the same points whose line integrals have different values).