MATH 18.02A - Problem Set 5 Solutions

Spring 2007

Part I - Problem 3D-9.

Recall that $\det A \cdot \det B = \det A \cdot B$ for matrices A and B with the same dimensions. Thus

$$\begin{aligned}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \cdot \left|\frac{\partial(u,v)}{\partial(x,y)}\right| &= \det \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \cdot \det \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= \det \begin{vmatrix} x_u u_x + x_v v_x & x_u u_y + x_v v_y \\ y_u u_x + y_v v_x & y_u u_y + y_v v_y \end{vmatrix}.\end{aligned}$$

However, keeping in mind that the pair (x, y) may be viewed as both independent variables and also as functions of (u, v) (and vice versa), the multivariable chain rule implies that

$$\begin{aligned}1 &= \frac{d}{dx}\,x = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial x}, \\ 0 &= \frac{d}{dx}\,y = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial x}.\end{aligned}$$

This shows that the first column of the Jacobian is $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and the second column can be accounted for similarly, showing that the overall matrix is $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.

Part II - Problem 1.

This integral has a relatively simple integrand, but it is over a very complicated region R_{xy}, which suggests that a change of variables should be used to simplify the domain. Set

$$u = \frac{x}{2} - y, \qquad v = 2x + y,$$

so that R_{uv} is now bounded by the equation $u^2 + v^2 = 1$, and the integrand can be rewritten as $x = \frac{2}{5}(x + y)$. The Jacobian is

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}^{-1} = \begin{vmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{vmatrix}^{-1} = \left(\frac{5}{2}\right)^{-1} = \frac{2}{5},$$

so the integral becomes

$$\begin{aligned}\iint_{R_{xy}} x\, dA &= \int_{R_{uv}} x(u,v) \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} dudv \\ &= \iint_{R_{uv}} \frac{2}{5}(x+y)\cdot\frac{2}{5}\, dudv.\end{aligned}$$

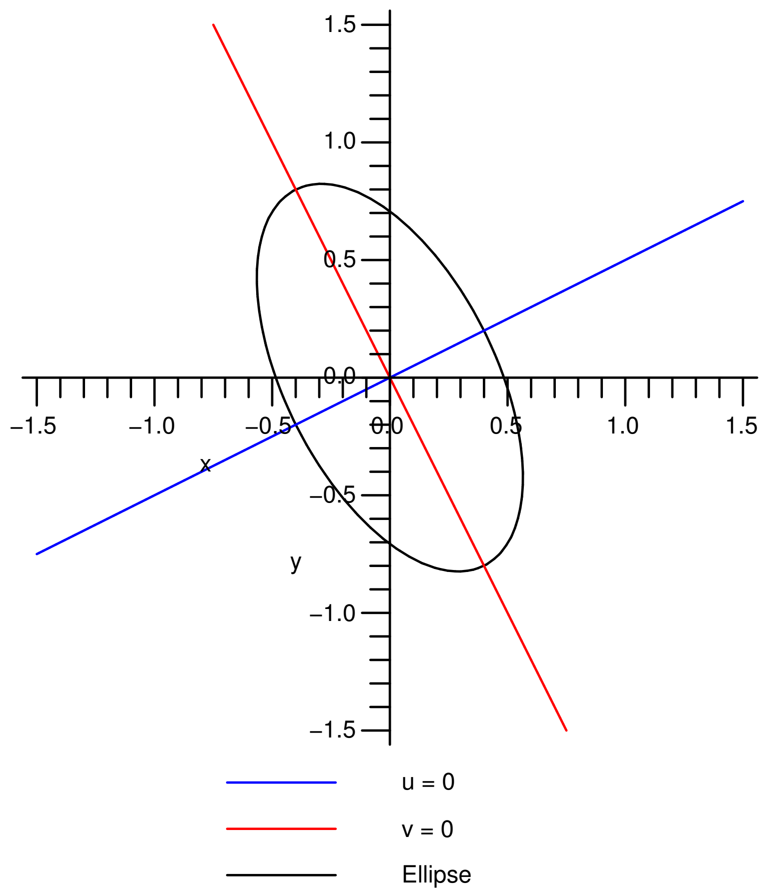
Finally, since R_{uv} is bounded by a circle of radius 1 in (u, v) coordinates, it makes sense to convert to polar coordinates

$$u = r\cos\theta, \qquad v = r\sin\theta.$$

The integral is now

$$\begin{aligned}&\frac{4}{25}\int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta) r\, drd\theta \\ &= \frac{4}{25}\int_0^{2\pi} (\cos\theta + \sin\theta)\frac{1}{3}\, d\theta = \boxed{\mathbf{0}},\end{aligned}$$

since both $\cos\theta$ and $\sin\theta$ integrate to zero over a period of 2π.



Remark. Although it is not necessary to understand the geometry in order to evaluate the integral, it will help illuminate the previous work. The integral in this problem can be viewed as the moment M_x of the region R (i.e., the weighted sum of all x-values).

The accompanying graph shows the shape of the region, which can be understood by considering the two lines $u = 0$ and $v = 0$. As seen in the picture, the effect of the substitution is to rotate the coordinate axes, where R is merely an ellipse. However, this region is clearly symmetric about the origin in the (x, y) plane, so the average x value must be 0, as found above.

Part II - Problem 2.

a)The volume of the tent is easily computed with cylindrical coordinates:

$$
\begin{aligned}
V = \iiint_R dV &= \int_0^{2\pi} \int_0^R \int_0^{100e^{-r^2}} r \, dz dr d\theta \\
&= \int_0^{2\pi} \int_0^R 100re^{-r^2} \, dz dr d\theta \\
&= \int_0^{2\pi} -50e^{-r^2} \Big|_0^R \, d\theta = \boxed{100\pi \left(1 - e^{-R^2}\right)}.
\end{aligned}
$$

b) The average density will be the quotient of the total mass of smoke and the volume from part **a)**. The mass is again found with an integral in cylindrical coordinates:

$$
\begin{aligned}
M &= \iiint_R \delta\, dV = \int_0^{2\pi}\int_0^R\int_0^{100e^{-r^2}} r\alpha z\; dz dr d\theta \\
&= \alpha \int_0^{2\pi}\int_0^R \frac{rz^2}{2}\Big|_0^{100e^{-r^2}} \alpha z\; dr d\theta \\
&= \alpha \int_0^{2\pi}\int_0^R 5000 r e^{-2r^2}\; dz dr d\theta \\
&= \alpha \int_0^{2\pi} -1250 e^{-2r^2}\Big|_0^R\; d\theta \\
&= 2500\alpha\pi\left(1-e^{-2R^2}\right).
\end{aligned}
$$

Therefore, the average smoke density is

$$
\begin{aligned}
\frac{M}{V} &= \frac{2500\alpha\pi\left(1-e^{-2R^2}\right)}{100\pi\left(1-e^{-R^2}\right)} \\
&= \boxed{25\alpha\left(1+e^{-R^2}\right)}.
\end{aligned}
$$

Part II - Problem 3.

a) The equation of the ellipsoid is closely related to a sphere, and the change of variables

$$
X = \frac{x}{a}, \qquad Y = \frac{y}{b}, \qquad Z = \frac{z}{c}
$$

makes the relationship clear. The Jacobian here is very simple, as

$$
\begin{aligned}
\left|\frac{\partial(X,Y,Z)}{\partial(x,y,z)}\right|^{-1} &= \det\begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix}^{-1} \\
&= \det\begin{vmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{vmatrix}^{-1} = abc.
\end{aligned}
$$

Therefore the volume is

$$
\begin{aligned}
V = \iiint_{R_{xyz}} dV &= \iiint_{R_{XYZ}} \left|\frac{\partial(X,Y,Z)}{\partial(x,y,z)}\right|^{-1} dX dY dZ \\
&= \iiint_{R_{XYZ}} abc\; dX dY dZ,
\end{aligned}
$$

where the region R_{XYZ} is a sphere of radius 1. Therefore, converting to spherical coordinates in X, Y, and Z gives

$$
V = abc\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2 \sin\phi\; d\rho\, d\phi\, d\theta = \boxed{\frac{4\pi abc}{3}}.
$$

b) By symmetry, the average x and y values are both zero. Following the given suggestion, the z-moment for the half-ellipsoid centered at the origin (and truncated by $z \geq 0$) is

$$\begin{aligned}
M_z &= \iiint_{R_{xyz}} z\, dV = abc \iiint_{R_{XYZ}} cZ\, dX\, dY\, dZ \\
&= abc^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \cos\phi \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= abc^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \frac{1}{4} \cos\phi \sin\phi\, d\phi\, d\theta \\
&= \frac{abc^2}{4} \int_0^{2\pi} \frac{\sin^2\phi}{2} \Big|_0^{\frac{\pi}{2}} d\theta = \frac{abc^2}{4} \cdot \frac{1}{2} \cdot 2\pi \\
&= \frac{\pi abc^2}{4}.
\end{aligned}$$

Hence the average z-value for this half-ellipsoid is

$$\frac{M_z}{V} = \frac{\frac{\pi abc^2}{4}}{\frac{2\pi abc}{3}} = \frac{3c}{8},$$

and flipping this gives that the z-value for the solid in question is $c - \frac{3c}{8} = \frac{5c}{8}$. Ultimately, the center of mass is

$$\boxed{\left(0, 0, \frac{5c}{8}\right)}.$$

Part II - Problem 4.

a) The path can be parameterized by $\mathbf{c}(t) = (x(t), y(t)) = (t, 0)$ for $0 \leq t \leq 1$, so the line integral is

$$\begin{aligned}
\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(x(t), y(t)) \cdot (x'(t), y'(t))\, dt \\
&= \int_0^1 \left(\frac{2t}{1}, \frac{-2t^2}{1}\right) \cdot (1, 0)\, dt = \int_0^1 2t\, dt = \boxed{1}.
\end{aligned}$$

b) Now the path has two parts, which can be written as

$$\mathbf{c}_1(t) = \begin{cases} (t, t) & \text{for } 0 \leq t \leq 1, \\ (1, 2-t) & \text{for } 1 \leq t \leq 2. \end{cases}$$

The line integral is

$$\begin{aligned}
\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \frac{2t}{(1+t)^2} - \frac{2t^2}{(1+t)^3}\, dt + \int_1^2 \frac{2}{(3-t)^3}\, dt \\
&= 2\int_0^1 \frac{t}{(1+t)^3}\, dt + \frac{1}{(3-t)^2}\bigg|_1^2 \\
&= 2\int_0^1 \frac{t+1}{(1+t)^3} + \frac{-1}{(1+t)^3}\, dt + \frac{3}{4} \\
&= 2\left(\frac{-1}{1+t} + \frac{1}{2(1+t)^2}\,\bigg|_0^1 + \frac{3}{4}\right. \\
&= \boxed{1}.
\end{aligned}$$

c) Along the middle segment of this path, there is a constant value $y = -1$. This means that $\mathbf{F}$ is undefined along this segment, and thus the overall line integral is $\boxed{\textbf{undefined}.}$

Remark. The field $\mathbf{F}$ is conservative in its domain, but part **c)** illustrates the fact that path independence only applies to paths that stay within the definition of $\mathbf{F}$.

Part II - Problem 5.

The striking feature of this vector field is that the $\hat{\mathbf{j}}$ component depends on the x coordinate, and so the effect of $\mathbf{F}$ on movement in the y-direction will depend on the x position. A simple example to test this is the two paths between $P_0 = (0,0)$ and $P_1 = (1,1)$ that travel in the coordinate directions in different orders.

Specifically, define

$$\mathbf{c}_1(t) = \begin{cases} (t,0) & \text{for} \quad 0 \le t \le 1, \\ (1, t-1) & \text{for} \quad 1 \le t \le 2, \end{cases}$$

and

$$\mathbf{c}_2(t) = \begin{cases} (0,t) & \text{for} \quad 0 \le t \le 1, \\ (t-1, 1) & \text{for} \quad 1 \le t \le 2. \end{cases}$$

Now the line integrals along these two paths evaluate to

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dt + \int_1^2 1\, dt = 1$$

and

$$\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dt + \int_1^2 0\, dt = 0,$$

which are unequal, and thus the line integral is not path independent.