MATH 18.02A - Problem Set 6 Solutions
Spring 2007

Part II - Problem 1.
a) The algebraic method begins with the observation that

$$\frac{\partial}{\partial x} f(x,y,z) = 2xyz + 2y^2 \cos 2x,$$

so

$$f(x,y,z) = \int 2xyz + 2y^2 \cos 2x \, dx = x^2yz + y^2 \sin 2x + g(y,z),$$

where $g(y,z)$ is for the moment considered an arbitary function of y and z. However,

$$\frac{\partial}{\partial y} f(x,y,z) = x^2 z + 2y \sin 2x,$$

and comparing this with the derivative $\partial\partial y$ of the earlier expression shows that $\frac{\partial}{\partial y} g(y,z) = 0$. Thus $f(x,y,z) = x^2yz + y^2 \sin 2x + g(z)$. Finally, comparing derivatives with respect to z shows that $\frac{\partial}{\partial z} g(z) = e^z$, so

$$\boxed{\boldsymbol{f(x,y,z) = x^2yz + y^2 \sin 2x + e^z}}\,.$$

b) To find the potential using the integration method, evaluate the line integral

$$f(a,b,c) = \iiint_{P_0}^{(a,b,c)} \mathbf{F} \cdot d\mathbf{r}$$

for an inital point P_0 in the domain of the gradient field. This field is defined for all values of (x,y,z), so $(0,0,0)$ is a simple choice, and the integral can be evaluated along straight line segments parallel to the coordinate axes:

$$f(a,b,c) = \iiint_{(0,0,0)}^{(a,0,0)} \mathbf{F} \cdot d\mathbf{r} + \iiint_{(a,0,0)}^{(a,b,0)} \mathbf{F} \cdot d\mathbf{r} + \iiint_{(a,b,0)}^{(a,b,c)} \mathbf{F} \cdot d\mathbf{r}.$$

Pick parameterizations of unit speed and evaluate

$$\begin{aligned} f(a,b,c) &= \iiint_0^a 0 \, dx + \iiint_0^b 2y \sin 2a \, dy + \iiint_0^c a^2b + e^z \, dz \\ &= 0 + b^2 \sin 2a + a^2bc + e^c; \end{aligned}$$

replace (a,b,c) by (x,y,z) to find the same answer as in part a).

Part II - Problem 2.
a) The curl of the field is

$$\operatorname{curl} \mathbf{F} = \frac{\partial}{\partial x} x^b y^a - \frac{\partial}{\partial y} x^a y^b = \boxed{\boldsymbol{b(x^{b-1}y^a - x^a y^{b-1})}}\,.$$

b) It will be a gradient field whenever the curl is zero, which means that either $b = 0$, or $x^{b-1}y^a - x^a y^{b-1} = 0$. The only way this can be satisfied is if the exponents are the same in each term, so $\mathbf{F}$ is a gradient field exactly when $\boxed{\boldsymbol{b = a + 1}}$ or $\boxed{\boldsymbol{b = 0}}$.

c) There are three different cases. For the first case, consider pairs $(a, a+1)$ with $a \neq -1$. The field is then

$$\mathbf{F} = x^a y^{a+1} \, \hat{\mathbf{\imath}} + x^{a+1} y^a \, \hat{\mathbf{\jmath}}.$$

Using the algebraic technique, the potential function satisfies $\frac{\partial}{\partial x} f(x, y) = x^a y^{a+1}$, so

$$f(x,y) = \frac{1}{a+1}(xy)^{a+1} + g(y),$$

where $g(y)$ is some differentiable function of y. The other partial derivative implies that

$$\frac{\partial}{\partial y} f = x^{a+1} y^a + \frac{\partial}{\partial y} g(y) = x^a y^{a+1},$$

so $g(y)$ is just a constant.

The second case covers pairs $(a, 0)$, where again $a \neq -1$. Here $\mathbf{F} = x^a \,\hat{\imath} + y^a \,\hat{\jmath}$, and the potential function is $f(x,y) = \frac{1}{a+1}(x^{a+1} + y^{a+1})$.

Finally, when $(a,b) = (-1, 0)$, the field is $\mathbf{F} = \frac{1}{x}\,\hat{\imath} + \frac{1}{y}\,\hat{\jmath}$. A potential function in this case is $f(x,y) = \ln x + \ln y$.

Overall, the potential functions are described by

$$f(x,y) = \boxed{\begin{cases} \frac{1}{a+1}(xy)^{a+1} & \textbf{if } (a,b) = (a, a+1) \textbf{ and } a \neq -1, \\ \frac{1}{a+1}(x^{a+1} + y^{a+1}) & \textbf{if } (a,b) = (a, 0) \textbf{ and } a \neq -1, \\ \ln x + \ln y & \textbf{if } (a,b) = (-1, 0). \end{cases}}$$

Part II - Problem 3.

a) The Fundamental Theorem of Calculus for Line Integrals states that the integral of a gradient field is path-independent, and that the value is found by evaluating the difference of the potential function at the starting and ending points. Therefore

$$\int_{\mathbf{c}} \mathbf{F} \cdot dr = \int_{\mathbf{c}} \vec{\nabla}\left(x^4 + 2x^3 y - \frac{y^4}{2}\right) \cdot dr = x^4 + 2x^3 y - \frac{y^4}{2}\bigg|_{(-1,-1)}^{(0,0)} = \boxed{\frac{-5}{2}}.$$

b) The integral may also be evaluated by picking any path from $(-1,-1)$ to $(0,0)$; an obvious choice is the straight line $\mathbf{c}(t) = (t,t)$ for $-1 \leq t \leq 0$. In order to evaluate the integral, calculate the gradient $\mathbf{F} = (4x^3 + 6x^2 y)\,\hat{\imath} + (2x^3 - 2y^3)\,\hat{\jmath}$. Now

$$\int_{\mathbf{c}} \mathbf{F} \cdot dr = \int_{-1}^{0} (10t^3\,\hat{\imath} + 0\,\hat{\jmath}) \cdot (\hat{\imath} + \hat{\jmath})\, dt = \int_{-1}^{0} 10t^3\, dt = \boxed{\frac{-5}{2}}.$$

Part II - Problem 4. Green's Theorem replaces a line integral of a vector field around a closed path by a double integral of the curl over the interior of the path. In this problem, the interior of the path is the triangle, and the curl is

$$\begin{aligned} \text{curl} &= \frac{\partial}{\partial x}\left(\frac{x^4 y}{2} + \frac{x^2}{2}\sin y - xy\right) - \frac{\partial}{\partial y}\left(2x^3 - x\cos y\right) \\ &= 2x^3 y + x\sin y - y - x\sin y = 2x^3 y - y. \end{aligned}$$

So the line integral is equivalent to the double integral $\iint_R -y\, dA$, where R is the triangle. Since R is symmetric about the y-axis, and that the integrand $2x^3 y$ is an odd function with respect to x, $\iint_R 2x^3 y\, dA = 0$. Furthermore, $-y$ is an even function with respect to x, so by symmetry the integral is just twice the integral over the triangle in the first quadrant bounded by $(0,0), (0,1)$, and $(2,0)$. This evaluates as

$$2\int_0^1 \int_0^{2-2y} -y\, dx dy = 2\int_0^1 -2y + 2y^2\, dy = -2 + \frac{4}{3} = \boxed{\frac{-2}{3}}.$$