18.02A Problem Set 8 (and revised Syllabus) - Spring 2007

Due Friday, Mar. 9 at 2:00

Part I (10 points)

Lecture 51. (*Fri., Mar. 2*) 3-dimensional curl; Conservative fields; Stokes' Theorem.
Read: Notes V12, V13 *Work:* 6E-3, 4; 6F-1b

Lecture 52. (*Tues., Mar. 6*) 3-dimensional flux; Stokes' Theorem (continued.)
Read: Notes V9 *Work:* 6F-2, 5

Lecture 53. (*Thurs., Mar. 8*) Divergence Theorem.
Read: Notes V9, V10 *Work:* 6C-3, 5, 6

The problems listed for the remainder of the lectures are only suggestions, and are not to be handed in.

Lecture 54. (*Fri., Mar. 9*) Applications of Divergence Theorem.
Read: Notes V15 *Work:* 6C-10; 6H-1, 2a

Lecture 55. (*Tues., Mar. 13*) More applications of Stokes' Theorem; Topology.
Read: Notes V14 *Work:* 6H-2b, 3

Lecture 56. (*Thurs., Mar. 15*) Review and summary of vector calculus.

Lecture 57. (*Fri., Mar. 16*) **Final Exam** covering lectures 40–55; Room 4-159, 2:00 – 4:00.

Part II (15 points)

Try each problem alone for 15 minutes before collaborating, and write up solutions independently. The problems are given in order according to the lecture schedule above.

Problem 1. (*5 pts: 2+2+1*) Let R be the first-quadrant region bounded by the sphere of radius 2. Its boundary contains four surfaces: S_{xy}, S_{xz}, and S_{yz} lie in their respective coordinate planes, while S_s is a portion of the sphere; orient each surface to point outward. The surfaces intersect in 6 different curves: $C_x, C_y, C_z, C_{xy}, C_{xz}$, and C_{yz}, where each curve lies either along a coordinate axis or plane. Let

Consider the flow field

$$\mathbf{F} = (z-y)\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + -x\,\hat{\mathbf{k}}.$$

a) Calculate $\iint_S \vec{\nabla} \times \mathbf{F} \cdot d\mathbf{S}$ for each of the surfaces.
b) Compute the positively oriented line integral $\iint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$ around each surface, and verify Stokes' Theorem as you go.
c) Use the arithmetic (i.e. addition laws) of line integrals to explain why the sum of the four surface integrals is zero.

Problem 2. (*2 pts*) Let $\mathbf{F} = (3\sin y + az^2)\,\hat{\mathbf{i}} + (3x\cos y - bze^{yz})\,\hat{\mathbf{j}} + (bxz - 2ye^{yz})\,\hat{\mathbf{k}}$. Find the value(s) of a and b such that $\mathbf{F}$ is conservative.

Problem 3. (*3 pts*) Suppose that $\mathbf{F} = (-yz + z^2)\,\hat{\mathbf{i}} + (xz + 3x)\,\hat{\mathbf{j}} + 2xz\,\hat{\mathbf{k}}$, and that S is a cylindrical shell of radius a centered at the origin that ranges from $z = 0$ to $z = 1$. Let $\mathbf{c}_0$ and

$\mathbf{c}_1$ denote the positively oriented bounding circles at $z = 0$ and $z = 1$ respectively. Calculate $\oint_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r}$ by using Stokes' Theorem for multi-connected regions. (*Hint: If S is oriented so that the normals all point in toward the z-axis, then the total boundary of S is $\partial S = \mathbf{c}_1 - \mathbf{c}_0$.*)

Problem 4. (*2 pts: 1+1*) a) Show that $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ for any vector field $\mathbf{F}$ with continuous second derivatives.
b) Let S be a closed surface. Use the Divergence Theorem and part a) to prove that $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0$.

Problem 5. (*3 pts*) Verify the Divergence Theorem for the field $\mathbf{F} = 2x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} - z\,\hat{\mathbf{k}}$ over the region R bounded by $z = 0, z = 1$ and $\left(\frac{x}{2}\right)^2 + y^2 = 1$.
Hint: The ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ can be parameterized by $x = a\cos\theta, y = b\sin\theta$.