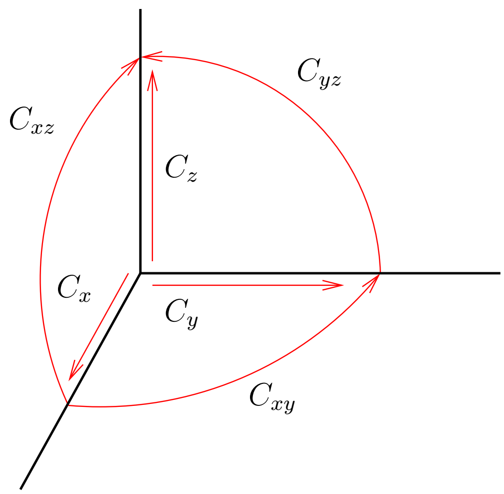
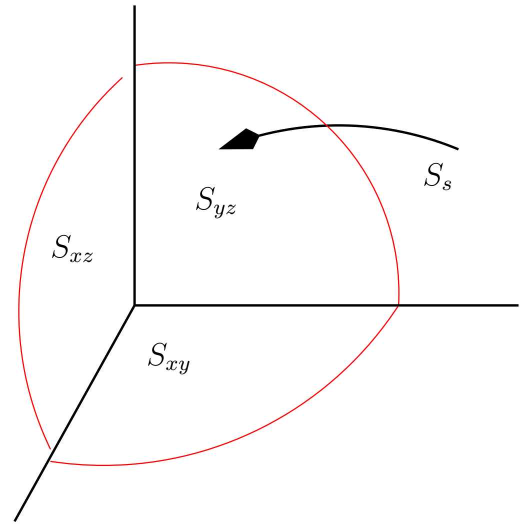
MATH 18.02A - Problem Set 8 Solutions
Spring 2007

Part II - Problem 1.

The curves and regions are labeled as shown in the following figures; the curves are oriented according to the parameterizations chosen in the solution to part b).





This problem requires frequent use of the integrals

$$\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi}{4}, \qquad \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta = \frac{1}{2}.$$

a) First, find the curl of $\mathbf{F}$:

$$\vec{\nabla}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z-y & 2y & -x \end{vmatrix} = \hat{\mathbf{i}}\cdot 0 - \hat{\mathbf{j}}\,(-1-1) + \hat{\mathbf{k}}\cdot 1 = 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

The normal differential vectors for S_{xy}, S_{xz} and S_{yz} are $-\hat{\mathbf{k}}, -\hat{\mathbf{j}}$ and $-\hat{\mathbf{i}}$, respectively, and each region is a quarter-circle with area π. Therefore

$$\iint_{S_{xy}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_{xy}} (2\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(-\hat{\mathbf{k}})\, dA = \boxed{-\boldsymbol{\pi}}\,,$$
$$\iint_{S_{xz}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_{xz}} (2\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(-\hat{\mathbf{j}})\, dA = \boxed{-\mathbf{2}\boldsymbol{\pi}}\,,$$
$$\iint_{S_{yz}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_{yz}} (2\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(-\hat{\mathbf{i}})\, dA = \boxed{\mathbf{0}}\,.$$

Finally, the outward normal on the surface of the sphere is

$$\hat{\mathbf{n}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{2}.$$

Using spherical coordinates $x = 2\sin\phi\cos\theta, y = 2\sin\phi\sin\theta, z = 2\cos\phi$,

$$\begin{aligned}
&\iint_{S_s} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} \\
&\quad = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (2\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot\left(\sin\phi\cos\theta\,\hat{\mathbf{\imath}}+\sin\phi\sin\theta\,\hat{\mathbf{j}}+\cos\phi\,\hat{\mathbf{k}}\right)\,4\sin\phi\,d\phi d\theta \\
&\quad = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 8\sin^2\phi\sin\theta + 4\sin\phi\cos\phi\,d\phi d\theta \\
&\quad = \int_0^{\frac{\pi}{2}} 2\pi\sin\theta + 2\,d\theta = 2\pi+\pi = \boxed{\mathbf{3\pi}}\,.
\end{aligned}$$

Note that $\iint_{S_{xy}+S_{xz}+S_{yz}+S_s} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.

b) The line integral $\oint_{\mathbf{c}} (z-y)\,dx + 2y\,dy - x\,dk$ must be computed over each of the paths. The integrals along the coordinate axes are very straightforward:

$$C_x : \int_0^2 0\,dx = \boxed{\mathbf{0}}\,, \qquad C_y : \int_0^2 2y\,dy = \boxed{\mathbf{4}}\,,$$

$$C_z : \int_0^2 0\,dz = \boxed{\mathbf{0}}\,.$$

For the other three paths, use the following parameterizations:

C_{xy} :	$x = 2\cos\theta$,	$y = 2\sin\theta$,	$z = 0$,
C_{xz} :	$x = 2\cos\theta$,	$y = 0$,	$z = 2\sin\theta$,
C_{yz} :	$x = 0$,	$y = 2\cos\theta$,	$z = 2\sin\theta$.

Now compute

$$\begin{aligned}
\oint_{C_{xy}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} (0-2\sin\theta)\cdot(-2\sin\theta) + 4\sin\theta\cdot 2\cos\theta\,d\theta = \boxed{\boldsymbol{\pi+4}}\,,\\
\oint_{C_{xz}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} (2\sin\theta-0)\cdot(-2\sin\theta) - 2\cos\theta\cdot 2\cos\theta\,d\theta = \boxed{\boldsymbol{-2\pi}}\,,\\
\oint_{C_{yz}} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{\frac{\pi}{2}} 4\cos\theta\cdot(-2\sin\theta) - 0\,d\theta = \boxed{\mathbf{-4}}\,.
\end{aligned}$$

It is easily verified that the sum of these integrals around the boundary of each surface gives the same result as in part bf a).

c) By Stokes' Theorem, each surface integral is equal to the line integral around its boundary. Therefore

$$\iint_{S_{xy}+S_{xz}+S_{yz}+S_s} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_{\partial S_{xy}+\partial S_{xz}+\partial S_{yz}+\partial S_s} \mathbf{F}\cdot d\mathbf{r}$$

and

$$\begin{aligned}
&\partial S_{xy}+\partial S_{xz}+\partial S_{yz}+\partial S_s \\
&\quad = (-C_x+C_y-C_{xy}) + (-C_z+C_x+C_{xz}) + (-C_y+C_z-C_{yz}) + (C_{xy}+C_{yz}-C_{xz}) = 0.
\end{aligned}$$

Remark. In general, if S is a closed surface that has been decomposed into simple, connected pieces $S_1, \ldots, S_n$, then this same sort of cancellation among the sum of the boundaries will occur. In other words, the "total" boundary of a closed surface is always 0 (i.e., there is no boundary).

Part II - Problem 2.

Calculate the curl:

$$\vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3\sin y + az^2 & 3x\cos y - bze^{yz} & bxz - 2ye^{yz} \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left(-2e^{yz} - 2yze^{yz} - (-bze^{yz} - bzye^{yz})\right) - \hat{\mathbf{j}}\,(bz - 2az) + \hat{\mathbf{k}}\,(3\cos y - 3\cos y)$$

$$= \hat{\mathbf{i}}\, e^{yz}(b-2)(1+yz) - \hat{\mathbf{j}}\, z(b-2a).$$

The field is conservative if and only if the curl is zero, which implies that $\boxed{\boldsymbol{a = 1, b = 2}}$.

Part II - Problem 3.

If S is oriented as suggested, then Stokes' Theorem implies that $\oint_{\mathbf{c}_1 - \mathbf{c}_0} \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA$, so the line integral around $\mathbf{c}_1$ can be calculated indirectly as

$$\oint_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathbf{c}_0} \mathbf{F} \cdot d\mathbf{r} + \iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA.$$

First, parameterize $\mathbf{c}_0$ by $x = a\cos\theta, y = a\sin\theta, z = 0$ and evaluate

$$\oint_{\mathbf{c}_0} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathbf{c}_0} (-yz + z^2)\, dx + (xz + 3x)\, dy + 2xz\, dz$$

$$= \int_0^{2\pi} 0 + 3a\cos\theta \cdot a\cos\theta + 0\, d\theta = 3\pi a^2.$$

Next, calculate the curl of $\mathbf{F}$:

$$\vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz + z^2 & xz + 3x & 2xz \end{vmatrix} = \hat{\mathbf{i}}\,(0 - x) - \hat{\mathbf{j}}\,(2z - 2z + y) + \hat{\mathbf{k}}\,(z + 3 + z)$$

$$= -x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + (2z+3)\,\hat{\mathbf{k}}.$$

It is unnecessary to parameterize S, since it is clear geometrically that the unit normal from a cylindrical shell of radius a is

$$\hat{\mathbf{n}} = \frac{-x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{a},$$

and thus the surface integral becomes

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{n}}\, dA = \iint_S (-x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}} + (2z+3)\,\hat{\mathbf{k}}) \cdot \left(\frac{-x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{a}\right) dA$$

$$= \iint_S \frac{x^2 + y^2}{a}\, dA = \iint_S a\, dA = a \cdot 2\pi a = 2\pi a^2.$$

Combining the two calculations gives $\oint_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{r} = 3\pi a^2 + 2\pi a^2 = \boxed{\boldsymbol{5\pi a^2}}$.

Part II - Problem 4.

a) Write $\mathbf{F} = M_1(x,y,z)\,\hat{\imath} + M_2(x,y,z)\,\hat{\jmath} + M_3(x,y,z)\,\hat{\mathbf{k}}$, and use the notation $\partial_{xy} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$. Now

$$\begin{aligned} \text{div}\,(\text{curl}\,\mathbf{F}) &= \text{div}\,\Big[(\partial_y M_3 - \partial_z M_2)\,\hat{\imath} + (\partial_z M_1 - \partial_x M_3)\,\hat{\jmath} + (\partial_x M_2 - \partial_y M_3)\,\hat{\mathbf{k}}\Big] \\ &= \partial_{yx} M_3 - \partial_{zx} M_2 + \partial_{zy} M_1 - \partial_{xy} M_3 + \partial_{xz} M_2 - \partial_{yz} M_3 = \boxed{\mathbf{0}}\,, \end{aligned}$$

by the equality of mixed partials.

b) Suppose that S is the boundary of a region R. The Divergence Theorem implies that

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iiint_R \text{div}\,(\text{curl}\,\mathbf{F})\,dV = \boxed{\mathbf{0}}\,.$$

Remark. Physical interpretation: The surface integral of curl$\mathbf{F}$ around a closed surface "measures" the total rotational effect of $\mathbf{F}$ on the surface (viewing the surface from its exterior, positive values correspond to clockwise rotation). But on a closed surface, all of the motion must balance out to zero, as there is no boundary curve on which excess rotation can "collect."

Part II - Problem 5.

First, the divergence is div $\mathbf{F} = 2 + 1 - 1 = 2$, so the triple integral over R is

$$\iiint_R \text{div}\,\mathbf{F}\,dV = \iiint_R 2\,dV = 2\cdot 2\pi = \boxed{\mathbf{4\pi}}\,,$$

since the area of an ellipse with radii a and b is πab.

The boundary of the region R has three surfaces – let S_1 and S_0 denote the top and bottom ellipses (at $z = 1$ and $z = 0$, respectively), and let S be the outer wall. On S_1 and S_0, the normal vectors are $\hat{\mathbf{n}} = \hat{\mathbf{k}}$ and $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, respectively, so

$$\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \iint_{S_1} -1\,dA = -2\pi, \quad \iint_{S_0} \mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \iint_{S_1} 0\,dA = 0.$$

Parameterize S by

$$\begin{gathered} x = 2\cos\theta, \qquad y = \sin\theta, \qquad z = z; \\ 0 \le \theta \le 2\pi, \qquad 0 \le z \le 1. \end{gathered}$$

The normal differential around S is

$$\mathbf{T}_\theta \times \mathbf{T}_z = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}} \\ -2\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath},$$

so

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^1 (4\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} - z\,\hat{\mathbf{k}})\cdot(\cos\theta\,\hat{\imath} + 2\sin\theta\,\hat{\jmath})\,dz d\theta \\ &= \int_0^{2\pi}\int_0^1 4\cos^2\theta + 2\sin^2\theta\,dz d\theta = 6\pi. \end{aligned}$$

Therefore, the total surface flux is $\iint_{S_1+S_0+S} \mathbf{F}\cdot\hat{\mathbf{n}}\,dA = -2\pi + 6\pi = \boxed{\mathbf{4\pi}}\,.$