Problem Solving Seminar - Fall 2012
Nov. 26

Reminder: The Putnam Exam will be given on Saturday, Dec. 1 from 9:00 A.M. - 5:00 P.M. in Lockett B9.

1. [**1995 A-1**] Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

2. [**1986 A-2**] What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$$

3. [**1996 B-3**] Given that $\{x_1, x_2, \ldots, x_n\} = \{1, 2, \ldots, n\}$, find, with proof, the largest possible value, as a function of n (with $n \geq 2$), of

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

4. [**2005 B-4**] For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, ..., x_n)$ of integers such that $|x_1| + |x_2| + + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

5. [**1997 A-5**] Let N_n denote the number of ordered n-tuples of positive integers $(a_1, a_2, \ldots, a_n)$ such that $1/a_1 + 1/a_2 + \cdots + 1/a_n = 1$. Determine whether N_{10} is even or odd.

6. [**2000 A-6**] Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$, then either $a_1 = 0$ or $a_2 = 0$.

For a bit of variety...

7. What is the smallest natural number that leaves remainders $1, 2, 3, 4, 5, 6, 7, 8$ and 9 when divided by $2, 3, 4, 5, 6, 7, 8, 9$ and 10, respectively?