Problem Solving Seminar - Fall 2012
Oct. 1

1. Prove that at some point in history, there was a single second in which at least two human beings were born. Note that this is never true of twins, even those delivered by Caesarian section!

2. A pair of integers m and n are said to have a *common divisor* d if both m and n are multiples of d. The integers are *relatively prime* if their only common divisor is 1.

   (a) Find a collection of 50 integers from 1 to 100 such that any pair of have a common divisor $d > 1$.

   (b) Prove that there are two relatively prime integers in any collection of $N + 1$ integers chosen from the range $[1, 2N]$.

3. A sequence of real numbers $\{a_i\}_{i=1}^N$ is *monotonically increasing* if $a_i \leq a_{i+1}$ for all i, and *monotonically decreasing* if $a_i \geq a_{i+1}$. A *subsequence* has the form $\{a_{i_k}\}_{k=1}^K$, where the indices i_k are strictly increasing. Prove that any sequence of length $N^2 + 1$ must have a monotonically increasing or decreasing subsequence of length $N + 1$ or more.

4. A rectangular chocolate bar is segmented into small squares, with dimensions m squares by n squares. The bar may be broken along any horizontal and vertical line between the squares, resulting in two smaller rectangles. Each of those smaller pieces can again be broken, continuing until all of the small squares have been separated from each other.

   (a) If $n = 1$, how many breaks are necessary to separate all of the squares?

   (b) How many breaks are necessary to separate the squares in the general case $m \times n$?

   (c) A family of rats have eaten several of the squares, leaving a single irregularly shaped piece that is composed of s connected squares. How many breaks are necessary to separate all of the squares in this piece?

5. **[1992 B2]** For nonnegative integers n and k, define $Q(n, k)$ to be the coefficient of x^k in the expansion of $(1 + x + x^2 + x^3)^n$. Prove that

$$Q(n, k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k - 2j}.$$

6. **[2010 B3]** There are 2010 boxes labeled $B_1, B_2, \ldots, B_{2010}$, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving exactly i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

7. **[2000 B6]** Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(1, 1, ..., 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.