Problem Solving Seminar - Fall 2012
Oct. 8

1. (a) The circle $x^2 + y^2 = 1$ contains many points with rational coefficients, such as $(1, 0)$ and $(\frac{3}{5}, \frac{4}{5})$. Does it contain infinitely many such points?
   (b) Prove that the circle $x^2 + y^2 = 3$ contains no points with rational coefficients.

2. What is the remainder when 2010^{2011} is divided by 2012?

3. (a) Find a collection of 100 integers between 1 and 200 such that none of the chosen integers is a multiple of any of the others.
   (b) Prove that any collection of 101 integers between 1 and 200 must contain a pair of integers such that one is a multiple of the other.

4. A *composition* of size n is a sequence of positive integers that sum to n. For example, the compositions of size 3 are

$$3, \quad 2+1, \quad 1+2, \quad 1+1+1.$$

   How many compositions of size n are there?

5. **[1988 B-1]** A composite integer is a product ab with a and b not necessarily distinct integers in $\{2, 3, 4, \dots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with x, y, z positive integers.

6. **[1989 A-1]** How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?
   Hint: 101 *is prime, but* 10101 *is not....*