Problem Solving Seminar - Fall 2012
Oct. 29

1. The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

   (a) Prove that

$$F_n = F_{n-1} + F_{n-3} + F_{n-5} \cdots + \begin{cases} F_1 & \text{if n is even,} \\ F_2 & \text{if n is odd.} \end{cases}$$

   (b) Prove that for $n \geq 1$,

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

   Hint: There is an easy geometric proof...

   (c) Let A_n denote the number of ways of tiling a $2 \times n$ rectangle with dominoes of either orientation (i.e., 1×2 and 2×1 rectangles). Find a formula for A_n in terms of Fibonacci numbers.

2. Without using a calculator, determine which quantity is larger:

$$\sqrt{20} + \sqrt{79} \quad \text{or} \quad \sqrt{19} + \sqrt{80}?$$

3. Suppose that n lines are drawn in the plane. This separates the plane into some number of distinct regions, whose boundaries are determined by the lines.

   (a) What is the minimum number of regions defined by n lines?

   (b) What is the maximum number of regions?

4. Evaluate $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$.
   Hint: Consider the sequence defined by $a_1 = \sqrt{2}$ *and* $a_n = \sqrt{2 + a_{n-1}}$ *for* $n \geq 2$.

5. **[2000 A-1]** Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

6. **[2004 A-3]** Define a sequence $\{u_n\}_{n=0}^{\infty}$ by by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

   for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)