Problem Solving Seminar - Fall 2012
Nov. 5

1. (a) Which number is bigger: e^π or π^e?
   (b) If $0 < a < b$, which is bigger: a^b or b^a?

2. Prove the following inequalities:
   (a) $1 + x \geq 2\sqrt{x}$ for $x \geq 0$,
   (b) $x + 1/x \geq 2$ for $x > 0$,
   (c) $2(x^2 + y^2) \geq (x + y)^2$, $x, y \in R$,
   (d) Prove that $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$, $x, y > 0$.

3. Show that $(a^2 + b^2)(a^4 + b^4) \geq (a^3 + b^3)^2$ for all a, b.

4. Show that $a^3 + b^3 \geq a^2b + ab^2$ for all a, b satisfying $a + b \geq 0$.

5. Suppose that $a, b, c \geq 0$. Prove the following inequalities:
   (a) $(a + b)(b + c)(c + a) \geq 8abc$,
   (b) $ab + bc + ca \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ab}$.

6. Suppose that the sequence $a_1, a_2, \ldots, a_n$ consists of the values $1, 4, 9, \ldots, n^2$ in some order. What is the maximum value of

$$a_1 + 2a_2 + 3a_3 + \cdots + na_n?$$

   What is the minimum value?

7. Prove that if $x \geq 0$, then $3x^3 - 6x^2 + 4 \geq 0$.

8. Let $x, y, z > 0$ with $x^4 + y^4 + z^4 = 27$. Prove that $x + y + z \leq 3\sqrt{3}$.

9. **[2002 B-3]** Show that, for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

10. **[2003 A-2]** Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1a_2 \cdots a_n)^{1/n} + (b_1b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$