Problem Solving Seminar - Fall 2013
Aug. 28

1. (a) Show that among any 10 points in a 3×3 rectangle, there is a pair of points not more than $\sqrt{2}$ apart.
   (b) Show that among any 7 points in a 3×4 rectangle there is a pair of points not more than $\sqrt{5}$ apart.
   (c) Show that among any 6 points in a 3×4 rectangle there is a pair of points not more than $\sqrt{5}$ apart.

2. (a) Show that among any 5 integers there is a pair whose difference is a multiple of 4.
   (b) Show that any odd integer n not divisible by 5 divides a number consisting of all 1s. For instance, 3 divides 111 and 7 divides 111,111.

3. Suppose that n is a positive integer and $x > 0$. Prove that
$$\frac{x^n}{(x+1)^{n+1}} \leq \frac{n^n}{(n+1)^{n+1}}.$$

4. You are given a collection of long fuses that each take exactly one hour to burn from end to end, although the speed of the flame is not necessarily the same throughout the rope.
   (a) How can you measure exactly 30 minutes using one rope?
   (b) How can you measure exactly 45 minutes using two ropes?
   (c) Is it possible to measure exactly 15 minutes using one rope?

5. A hiker spends all day climbing a mountain and camps overnight. She then spends the following day hiking back down. Show that there is some time where she is at exactly the same point on the trail both days.

6. **[2002 A2]** Given any five points on a sphere, show that at least four of them must lie on some closed hemisphere.