Problem Solving Seminar - Fall 2013
Oct. 30

1. For each of the following pairs, determine which expression is larger:
   (a) 10^π or π^{10}?
   (b) 2013^{2013} or 2012^{2014}?
   (c) $2^{3^{4^5}}$ or $5^{4^{3^2}}$?
   (d) $\sqrt{2000}+\sqrt{13}$ or $\sqrt{1999}+\sqrt{14}$?

2. (a) Alice took 15 minutes to wash 5 windows, while Bob needed 60 minutes to wash 10 windows. How long will it take Alice and Bob working together to wash 30 windows?
   (b) What was the average amount of time per window with Alice and Bob working together?
   (c) The *harmonic mean* of two positive numbers a and b is
   $$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$
   What is the harmonic mean of 3 and 6? Verify that this is the same as your answer to part (b). How does this compare to the average of 3 and 6?

3. Suppose that $a, b, c \geq 0$.
   (a) Prove that $(a+b)(a+c)(b+c) > 8abc$.
   (b) Prove that $\left(a^2b+b^2c+c^2a\right)\left(ab^2+bc^2+ca^2\right) \geq 9a^2b^2c^2$.

4. (a) Prove that $n! > \left(\frac{n}{e}\right)^n$.
   Hint: Consider the Taylor series for e^x when $x = n$.
   (b) Prove that $n! < \left(\frac{n+1}{2}\right)^n$ for $n \geq 2$.
   Hint: What does the AGM inequality say when $a_1 = 1, a_2 = 2, \ldots, a_n = n$?

5. Suppose that $a_1, a_2, \ldots, a_n$ are the integers $1, 2, \ldots, n$ permuted in some order. What is the smallest possible value of
   $$\frac{a_1}{1}+\frac{a_2}{2}+\cdots+\frac{a_n}{n}?$$

6. If $x, y, z > 0$ and $x+y+z = 1$, find the minimum value of
   $$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$
   Is there a maximum value for this expression?

7. **[1996 B2]** Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

8. **[1978 A5]** Suppose that $x_1, \ldots, x_n$ are real numbers in the interval $(0, \pi)$, and let $x = \frac{x_1 + \cdots + x_n}{n}$. Prove that

$$\prod_{j=1}^{n} \left(\frac{\sin(x_j)}{x_j}\right) \leq \left(\frac{\sin(x)}{x}\right)^n.$$

Challenge.

1. The *harmonic, geometric,* and *arithmetic means* of positive numbers $a_1, \ldots, a_n$ are defined, respectively, as

$$\begin{aligned} H(a_1, \ldots, a_n) &:= \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}, \\ G(a_1, \ldots, a_n) &:= (a_1 \cdots a_n)^{\frac{1}{n}}, \\ A(a_1, \ldots, a_n) &:= \frac{a_1 + \cdots + a_n}{n}. \end{aligned}$$

Prove that

$$H(a_1, \ldots, a_n) \leq G(a_1, \ldots, a_n) \leq A(a_1, \ldots, a_n).$$