Problem Solving Seminar - Fall 2013
Nov. 6

1. (a) Prove that if a knight is placed on every square of an 8×8 chessboard, it is possible for every knight to move simultaneously.

   (b) If knights are placed on every square of a 7×7 chessboard, is it possible for each one to move simultaneously?

   Hint: If a knight is on a black square, where can it move?

2. (a) The numbers $1, 2, \ldots, 10$ are listed in order. You are allowed to switch the position of any two numbers in the list that have exactly one number in between them. For example, in the first move you could switch 5 and 7, resulting in

   $$1\ 2\ 3\ 4\ 7\ 6\ 5\ 8\ 9\ 10.$$

   Is there a sequence of moves that reverses the digits, so that they read $10, 9, \ldots, 1$?

   (b) Now you are allowed to "cyclically permute" any three adjacent digits, replacing *abc* by *cab*. For example, the first move applied to 567 gives

   $$1\ 2\ 3\ 4\ 7\ 5\ 6\ 8\ 9\ 10.$$

   Is there a sequence of moves that reverses the digits?

   (c) Repeat both parts beginning with the digits $1, 2, \ldots, 9$.

3. On a certain tropical island there are R red chameleons, B blue chameleons, and Y yellow chameleons. When two chameleons of different color meet, they both change to the third color.

   (a) Show that if $(R, B, Y) = (1, 1, 1)$, then it is possible for all three chameleons to end up the same color, but with the triple $(1, 2, 0)$ they can never be the same color.

   (b) Show that if there are four chameleons, then it is always possible for them to become the same color.

   (c) Find a simple general criterion that determines whether or not a given triple (R, B, Y) of chameleons can all becomes the same color. As test cases, your criterion should show that $(4, 6, 13)$ and $(6, 1, 0)$ can all become the same color, but $(12, 10, 8)$ cannot.

4. An infection spreads among the cells of an $n \times n$ checkerboard by the following rule: If at least two of the four adjacent cells of an uninfected cell are infected, then that cell becomes infected.

   (a) Show that if each of the n cells along a diagonal are infected, then the entire checkerboard will become infected.

   (b) Show that if fewer than n cells are infected initially, then it is impossible for the entire board to become infected.

5. **[1954 A1]** Let n be an odd integer. Let A be a symmetric matrix such that each row and column consists of some permutation of $\{1, 2, \ldots, n\}$. Show that each of the integers $1, 2, \ldots, n$ appears in the main diagonal.

   Recall that a matrix $\{a_{ij}\}$ is symmetric *if $a_{ij} = a_{ji}$ for all i, j.*

6. **[1955 A3]** Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of monotonically decreasing positive terms whose sum converges. Let S be the set of all $\sum_{n\geq 1} b_n$, where $\{b_n\}$ ranges over all subsequences of the a_n. Show that S is an interval if and only if

$$a_{n-1} < \sum_{i=n}^{\infty} a_i \qquad \text{for all } n.$$

Challenge.

1. You begin with a collection of 15 pebbles that are divided into some number of piles. Now remove one pebble from each pile, and place them all in a new pile. For example, if the pebbles are initially split into piles with $4, 7$, and 4 pebbles, then at the next step the piles have $3, 3$, and 6 pebbles, with a new pile of 3 pebbles.

   (a) Calculate the next few steps of the given example.

   (b) What happens in the long run as this process is iterated many times? How does your answer depend on the initial configuration?