Problem Solving Seminar - Fall 2013
Nov. 13

1. (a) Find a factorization of the polynomial $f(z) = z^4 + z^2 + 1$ into lower-degree polynomials with integer coefficients.

   (b) Determine all of the (complex) roots of the polynomial $g(z) = z^5 + z^3 + z^2 + 1$.

   Hint: What happens if you plug in $z = i$?

   (c) Let $h(z) = (z-1)^4 + (z+1)^4$. Does $h(z)$ have any real roots?

2. Let $m(x) = x^2 - 2$ and $n(x) = 2x^3 + x^2 - 10x - 5$, and observe that these polynomials have roots $m(\sqrt{2}) = n(\sqrt{5}) = 0$.

   (a) Find a polynomial $p(x)$ with integer coefficients that has the root $\sqrt{2} + \frac{1}{\sqrt{2}}$. Find a polynomial $q(x)$ that has the root $\frac{1}{\sqrt{5}}$.

   Hint: There is an easy way to do this that requires very little calculation.

   (b) Find a polynomial $r(x)$ that has the root $\sqrt{2} - \sqrt{5}$.

3. Show that the determinant of the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$$

   is always nonzero.

4. Consider polynomials of the form $f(z) = 1 + z^{n_1} + z^{n_2} + \cdots + z^{n_k}$, where $n_1 < n_2 < \cdots < n_k$ are positive integers.

   (a) Find the roots of $f(z) = 1 + z + z^2 + \cdots + z^n$ for $n = 1, 2, 3$. Show that all of these roots satisfy $|z| = 1$.

   (b) Show that $f(z) = 1 + z + z^3$ has a real root in the interval $(-1, 0)$.

   (c) Show that for any $f(z)$ of the given form, all roots satisfy

$$\frac{1}{2} \leq |z| \leq 2.$$

   Hint: If $|z| < \frac{1}{2}$, how does $|z^{n_1} + \cdots + z^{n_k}|$ compare to 1?

   (Challenge) Show that for any $f(z)$ of the given form, all roots satisfy

$$\frac{\sqrt{5}-1}{2} \leq |z| \leq \frac{\sqrt{5}+1}{2}.$$

5. **[1990 B3]** Let S be the set of 2×2 matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} < 200$. Show that if S has more than $50387 (= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

6. **[1992 B5]** Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the sequence $\{D_n/n!\}_{n\geq 2}$ bounded?

Challenge.

1. In this problem you will explore the game of *Chomp*. This two-player game begins with an $m \times n$ checkerboard, and on each turn a player selects a square and removes all other squares that are to its right or above. For example, on the first move of the game, the first player must remove some rectangle from the upper right corner of the board. The loser is whoever takes the final square in the lower left corner.

   (a) Show that on a $1 \times n$ board the first player can always win.

   (b) On a standard 8×8 board, is it better to be the first or second player?

   (c) Is there a general winning strategy on an $m \times n$ board?