Problem Solving Seminar - Fall 2013
Nov. 20

1. (a) Show that any positive integer can be represented as a sum of distinct powers of 2. In other words, a positive integer n can be written as

$$n = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_k},$$

where $m_1 > m_2 > \cdots > m_k$. Find these representations for $n = 8, 83, 160$ and 1023.

(b) Prove that any integer n can be written as

$$n = a_k 3^k + a_{k-1} 3^{k-1} + \cdots a_1 3 + a_0,$$

where $a_i = -1, 0$, or 1 for all i. Find these representations for $n = 8, 160, -444$ and 1094.

(c) Are these representations (from both parts) unique? Specifically, is it possible that

$$2^{m_1} + \cdots + 2^{m_k} = 2^{M_1} + \cdots + 2^{M_k}$$

if the set of m_i is different from the M_i? And is it possible that

$$a_k 3^k + \cdots + a_1 3 + a_0 = A_k 3^k + \cdots + A_1 3 + A_0$$

for distinct a_i, A_i?

2. Construct two lists by writing the powers of 10 in base 2 and base 5. For example, $10 = 1010_2$ and $10 = 20_5$. The first several terms in both series are

$$1010_2, 1100100_2, 1111101000_2, 10011100010000_2 \ldots$$
$$20_5, 400_5, 13000_5, 310000_5, \ldots$$

Consider the lengths of each entry in the two lists. In the first list, the terms have $4, 7, 10, 14, \ldots$ digits, and in the second list the terms have $2, 3, 5, 6, \ldots$ digits. Show that the digit lengths are distinct among the two sequences, and that furthermore, for any integer $n \geq 2$ there is exactly one term in one of the sequences with n digits. In which sequence is there a term with 20 digits?

Hint: If $\log_2(n) = 6.64$*, then how many digits are in the base 2 expansion of* n*?*

3. Suppose that $f(x)$ is a differentiable real function on $[0, \infty)$. Prove that if $f(0) \geq 0$ and $f'(x) > f(x)$, then $f(x)$ is always positive.

4. **[1990 B1]** Find all differentiable real functions $f(x)$ that satisfy

$$f(x) = \sqrt{2013 + \int_0^x \left(f(t)^2 + f'(t)^2\right) dt}.$$

5. **[1996 B1]** A subset X of the positive integers is *selfish* if $|X| \in X$, and is *minimal selfish* if it is no longer selfish when any element is removed. How many subsets of $\{1, 2, \ldots, N\}$ are minimal selfish?

6. **[2001 A1]** Consider a set S and a binary operation $*$ on S (that is, for each $a, b \in S$, $a * b \in S$). Assume that $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

7. **[1988 A5]** Prove that there is a unique function f that maps the positive reals to the positive reals, and also satisfies $f(f(x)) = 6x - f(x)$ for all $x > 0$.

Challenge.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function that satisfies

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

(a) Describe all continuous functions that satisfy the given equation.

(b) If f is not required to be continuous, are there other possibilities?

(c) Find all continuous functions that satisfy $f(xy) = xf(y) + yf(x)$.