# Problem Solving Seminar - Fall 2013
## Dec. 4

1. (a) Let $f$ be a real-valued function on the plane such that for every equilateral triangle $ABC$, $f(A) + f(B) + f(C) = 0$. Prove that $f(P) = 0$ for every point $P$.

   *Hint: Tile a hexagon with equilateral triangles.*

   (b) **[1986 A1]** Let $f$ be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D) = 0$. Does it follow that $f(P) = 0$ for all points $P$ in the plane?

2. (a) A *stair-step* path is made up of line segments on integer points $(x, y)$ that move either one unit right (to $(x + 1, y)$) or one unit up (to $(x, y + 1)$). How many stair-step paths are there from $(0, 0)$ to $(m, n)$?

   (b) **[2005 A2]** Let $S = \{(a, b)|a = 1, 2, \ldots, n, b = 1, 2, 3\}$. A *rook tour* of $S$ is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ is sequence such that

   (i) $p_i \in S$,

   (ii) $p_i$ and $p_{i+1}$ are a unit distance apart, for $1 \leq i < 3n$,

   (iii) for each $p \in S$ there is a unique $i$ such that $p_i = p$.

   How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

3. (a) Prove that $\sqrt{n+1} - \sqrt{n}$ is arbitrarily small as $n \to \infty$.

   (b) **[1990 A2]** Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, where $n$ and $m$ are integers?

4. (a) Show that the number of ways of writing $10 = a_1+a_2+a_3$, where $a_1 \geq a_2 \geq a_3 \geq 0$ are integers, is the same as the number of ways of writing $10 = b_1+b_2+b_3+b_4+b_5$, where $b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 \geq 0$ are integers. How does this generalize?

   (b) **[2005 B4]** For positive integers $m$ and $n$, let $f(m, n)$ denote the number of $n$-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

5. (a) Suppose that $P_1, P_2, \ldots, P_n$ are points in the plane, no three of which are collinear. Prove that there is a polygon with these points as its vertices.

   *Hint: Consider the path that traverses the points in an arbitrary order. If it is not a polygon, then there are two segments that cross each other. Can you construct a new path that removes this crossing?*

   (b) Suppose that $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ are points in the plane, no three of which are collinear. Prove that there are $n$ line segments that each join some $P_i$ and $Q_j$ such that none of the segments cross.

   (c) **[2005 B6]**

   Let $n \geq 4$, and suppose that $P_1, P_2, \ldots, P_n$ are randomly, uniformly and independently, chosen points on a circle. Consider the convex $n$-gon whose vertices are the $P_i$. What is the probability that at least one of the vertex angles is acute?

**Challenge.**

1. You are the last person in line to board a full airplane flight with 100 seats. Each person has an assigned seat, but the first person to board drops his boarding pass and takes a seat at random. Each subsequent passenger then takes his/her assigned seat, or if that seat is already taken, picks a random empty seat. What is probability that when you board the final empty seat is your assigned seat?