Problem Solving Seminar - Fall 2013
Sep. 4

1. (a) Prove that
$$1 + 3 + \cdots + (2n-1) = n^2.$$
   (b) Find (and prove) a formula for
$$1^3 + 2^3 + 3^3 + \cdots + n^3.$$

2. Show that every number in the sequence
$$1007, 10017, 100117, 1001117, \ldots$$
is divisible by 53.

   Hint: Can you write the n-th term in the sequence in terms of the $(n-1)$*st?*

3. (a) Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.

   Hint: First, find configurations with 6, 7, *and* 8 *squares. Then use the configuration with* 6 *squares to construct one with* 9.

   (b) Show that for any $n \geq 6$, an equilateral triangle can be dissected into n smaller equilateral triangles. Is there such a configuration with 5 triangles?

4. Prove that
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$
   Hint: Inequalities involving radicals can be easier if you square both sides.

5. Write $1, 1$ on one line. On the next line, write the sum of any two adjacent numbers between them, so the second line is $1, 2, 1$. Continue this process, so the third line is $1, 3, 2, 3, 1$, etc.

   (a) Write the first several lines.
   (b) Calculate the sum of the numbers in each line that you wrote.
   (c) What is the sum across the 10th line? The 100th?
   (d) What would happen if the first line began with $1, 2$ instead?

6. **[2005 A1]** Prove that every positive integer can be written as the sum of one or more integers of the form $2^s 3^t$, where s and t are nonnegative integers and no term in the sum is a multiple of another.

Challenge.

1. (a) Is there a circle in the plane that contains exactly 314 lattice points (points (a, b) where a and b are integers)?

   (*Optional*): Is there such a circle with radius exactly 10?

   (b) What is the largest possible number of lattice points that a circle of radius 10 can contain? The smallest? Find the best bounds that you are able.

   (c) Generalize your answers to circles of radius n.