Problem Solving Seminar - Fall 2013
Sep. 11

1. An $m \times n$ checkerboard is composed of 1×1 squares that alternate in color between red and black along each row and column.

   (a) Verify that a 3×2 checkerboard contains 18 distinct rectangles formed from the colored squares. How many rectangles are found in the 3×3 checkerboard?

   (b) How many of the rectangles in the 3×3 checkerboard have the same color in all four corners?

   (c) Generalize both problems to the $m \times n$ case.
   Hint: There is a "four-line" proof for enumerating the number of rectangles with no color restrictions.

2. An *n-tromino* is the result of removing a square of size n from the lower left corner of a square of size $2n$. In this problem you will explore the shapes that can be exactly covered by 1-trominoes.

   (a) Show that a 2-tromino can be tiled by 1-trominoes.

   (b) For $n = 3, 4, 5$, determine whether or not an n-tromino can be tiled by 1-trominoes.

   (c) Can a 2013-tromino be tiled by 1-trominoes?

   (d) Prove that if a 1×1 square is removed from the lower left of a $2^n \times 2^n$ square, the remaining figure can be tiled by 1-trominoes.
   Hint: Divide and conquer.

3. Arrange the integers $1, 2, \ldots, n$ consecutively around a circle. Remove the number 2 and proceed by removing every other number until only one remains. So, for example, if $n = 5$, then 2, 4, 1, and 5 are removed, with 3 remaining.

   (a) What is the last number remaining if $n = 2^p$?

   (b) What is the last number remaining if $n = 2^p + 1$?

(*Challenge*): What is the last number remaining for general n?

4. **[1985 A1]** Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that:

   (a) $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and

   (b) $A_1 \cap A_2 \cap A_3 = \emptyset$.

   Express your answer in the form $2^a 3^b 5^c 7^d$, where a, b, c, d are nonnegative integers.

5. **[2003 A1]** Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2+2, 1+1+2, 1+1+1+1$.

Challenge.

1. Given a natural number written in binary, you are allowed to insert addition signs between any adjacent digits, and then calculate the resulting sum. The process then repeats until 1 is reached. For example, the following is a legal sequence of moves:

$$\begin{aligned} 110011110101 &\to 11001111 + 0 + 101 = 11010100 \\ 11010100 &\to 110 + 101 + 00 = 1011 \\ 1011 &\to 101 + 1 = 110 \\ 110 &\to 1 + 1 + 0 = 10 \\ 10 &\to 1 + 0 = 1. \end{aligned}$$

   Your goal is to minimize the number of steps in the process.

   (a) The *naive* approach is to insert an addition sign between every pair of digits, for example:

   $$111 \to 1 + 1 + 1 = 11 \to 1 + 1 \to 10 \to 1 + 0 = 1.$$

   Prove that the naive approach is not optimal by finding a shorter sequence for 111.

   (b) Can you improve on the 5 moves used in the initial example?

   (c) Given a binary number with n digits, what is the minimal number of steps?
   Hint: The answer does not necessarily depend on n *...*