Problem Solving Seminar - Fall 2013
Sep. 18

1. (a) Evaluate the integral
$$\int_0^1 x^2 + \sqrt{x}\, dx.$$
(b) If $b > 0$, evaluate the integral
$$\int_0^1 x^b + x^{\frac{1}{b}}\, dx.$$
(c) Suppose that $f(x) : [0, a] \to [0, a]$ is a continuous, increasing function that satisfies $f(0) = 0$ and $f(a) = a$. Show that
$$\int_0^a f(x) + f^{-1}(x)\, dx = a^2.$$
Hint: Draw the right picture, and the problem becomes geometric ...
(d) Evaluate the integral
$$\int_0^1 2^{x^2} + \sqrt{\log_2(x+1)}\, dx.$$
Hint: Try adding a constant to the integrand.

2. You have 100 feet of fence with which to build a garden.
(a) Prove that the rectangular garden with the largest area is a square with side length 25.
(b) What is the shape of the largest triangular garden?

3. Suppose that $f(x)$ is a polynomial that satisfies $f(x) + f'(x) \geq 0$ for all x. For example, $x^2 + 1$ satisfies this condition. Is it possible for there to be a point c with $f(c) < 0$?

4. Suppose that $f(x)$ is a continuous real function and $f(x) \geq 0$ for all x.
(a) Show that if $\int_{-1}^1 x^2 f(x) dx = 0$, then $f(x) = 0$ on $[0, 1]$.
(b) Find an example of a non-zero function that satisfies $\int_{-1}^1 x f(x) = 0$.

5. **[1993 A1]** The horizontal line $L : y = c$ intersects the curve $C : y = 2x - 3x^3$ at two points in the first quadrant; call these P and Q. Let A be the region above P and Q bounded by L and C. Let B be the region bounded on the left by the y-axis, above by L, and to the right by C. Find the value of c such that the areas of A and B are the same.

6. **[1964 A2]** Let α be a real number. Find all continuous functions $f(x) : [0, 1] \to (0, \infty)$ such that $\int_0^1 f(x) dx = 1$, $\int_0^1 x f(x) dx = \alpha$ and $\int_0^1 x^2 f(x) dx = \alpha^2$.

Challenge.

1. Suppose that $f(x)$ is a continuous real function that satisfies $\int_0^1 x^n f(x)dx = 0$ for all integers $n \geq 0$. Is it true that $f(x) = 0$ on $[0, 1]$?