Problem Solving Seminar - Fall 2013
Oct. 2

1. (a) Prove that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges. Try to do this without using calculus.
   Hint: How many terms are at least $1/2$? How many are at least $1/4$?

   (b) Show that
$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1.$$

   (c) Evaluate
$$\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots .$$

2. Find a closed formula for $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$.

   Hint: Calculate a few examples and look for a recurrence.

3. (a) Evaluate $\sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots}}}$.
   Hint: Consider the sequence defined by $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$ for $n \geq 2$.

   (b) Determine whether it is possible to evaluate the following expression, and find its value if yes: $\sqrt{1 - \sqrt{1 - \sqrt{1 - \ldots}}}$.

   (c) Prove that the following expression is bounded for all N: $\sqrt{2\sqrt{3\sqrt{\ldots\sqrt{N}}}}$.

   (d) Determine whether the following expression is bounded for all N:
$$\sqrt{1 + \sqrt{2 + \sqrt{\ldots\sqrt{(N-1) + \sqrt{N}}}}}.$$

4. **[1951 A7]** Show that if $\sum_{n\geq 1} a_n$ converges, then so does $\sum_{n\geq 1} \frac{b_n}{n}$.

5. **[1984 A2]** Evaluate
$$\frac{6}{(9-4)(3-2)} + \frac{36}{(27-8)(9-4)} + \cdots + \frac{6^n}{(3^{n+1}-2^{n+1})(3^n-2^n)} + \cdots .$$

Challenge.

1. Show that the series

$$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \cdots + \frac{2^n}{1+x^{2^n}} + \cdots$$

converges when $x > 1$, and in this case find its sum.