

**Problem Solving Seminar - Fall 2013**  
**Oct. 9**

1. Given a 3-gallon jug and 5-gallon jug, it is possible to measure exactly 2 gallons as follows:
  - Fill the 5-gallon jug.
  - Pour the 5-gallon jug into the 3-gallon jug until it is full, leaving 2 gallons.
  - (a) Give a procedure for measuring exactly 4 gallons.
  - (b) For all volumes between 1 and 8 gallons, describe which can be measured exactly.
  - (c) Determine which one of the following is impossible:
    - Measure 10 gallons using 8 and 12-gallon jugs.
    - Measure 10 gallons using 8 and 13-gallon jugs.
  - (d) Generalize the result: if you are given  $n$  and  $m$ -gallon jugs, what volumes can be measured exactly?
2.
  - (a) How many distinct rotations are there of a cube? Equivalently, if a die is labeled in the standard way, in how many distinct ways may it be placed on a table in a fixed orientation?

*Hint: If you know the position of 1 and 2, is that sufficient to determine the remaining faces?*
  - (b) How many distinct rotations are there of a regular tetrahedron?
3. A Cop and Robber are positioned on adjacent vertices of a cube. Beginning with the Robber, the players alternate moves, with each being allowed to move along one edge per turn. If both players employ perfect strategy, is it possible for the Cop to catch the Robber?
4.
  - (a) You are given a cube of cheese that measures 3 inches on each side. Using a large flat knife, your task is to cut the cheese into smaller cubes that are 1 inch per edge. You are allowed to reposition the cheese so that the knife passes through multiple pieces with one cut. What is the minimum number of cuts required?

*Hint: Without moving any pieces, 6 cuts are enough. Can you do better?*
  - (b) What is the minimum number of cuts needed for a  $4 \times 4 \times 4$  cube?
5.
  - (a) A spider is at one ceiling corner of a room with dimensions  $10 \times 10 \times 10$ , and a fly is in the floor corner that is diagonally opposite. What is the shortest path that the spider can take along the ceiling, walls, and floor to reach the fly?
  - (b) What if the fly is in the middle of the floor?
  - (c) Redo parts a) and b) for a room that is  $10 \times 10 \times 20$ .
6. [1988 A4] The plane is divided into 3 disjoint sets. Can we always find two points in the same set a distance 1 apart? What about 9 disjoint sets?

**Challenge.**

1. (a) You are given 3 gold bars, of which one is counterfeit, and therefore lighter than the others. Using one weighing, identify the counterfeit bar.
- (b) Given 16 bars with one lighter counterfeit, identify it within 3 weighings.
- (c) Now suppose you are given 3 gold bars with one counterfeit, but you do not know if it is heavier or lighter than the authentic bars (it is definitely not exactly the same weight). How many weighings are needed to identify the counterfeit?
- (d) Given 12 bars and one counterfeit that may be lighter or heavier, identify it with 3 weighings.