Problem Solving Seminar - Fall 2013
Oct. 16

1. (a) Find a polynomial with integer coefficients such that $f(1) = 6, f(2) = 11$, and $f(3) = 22$.
   (b) Suppose that $g(x)$ is a polynomial with integer coefficients. Is it possible that $g(1) = 1$ and $g(2013) = 2012$?
   (c) If $h(x)$ is a polynomial that satisfies $h(1) = 1$ and $h(2) = 2$, is it possible for h to have **non**-integer coefficients?

2. Prove that the product of three consecutive integers can never be a perfect power (in other words, m^a for some integers m and a).

   Hint: Write the product as $(n-1)n(n+1)$. Can n and $n^2 - 1$ share any common factors?

3. One of the first things every child learns is how to count to 10 on her fingers. However, it is almost as easy to count to 25 by representing two-digit numbers in base 5. For example, holding up 3 fingers on the left hand and 4 on the right represents $34_5 = 3 \cdot 5 + 4 = 19$. By grouping fingers in other ways, what is the most efficient counting scheme that you can devise with 10 fingers?

4. (a) Show that the equation $x^2 - y^2 = a^3$ has positive integer solutions in x and y if a is any positive integer greater than 1.
   (b) Show that if $a = 3$, there is more than one solution.
   (c) For which values of a are the solutions unique?

5. Let R_n be the n-digit number $111 \cdots 1$.
   (a) For which n is R_n a multiple of 3?
   (b) For which n is R_n a multiple of 37?
   (c) For which n is R_n a multiple of 41?
   (d) For which n is R_n a multiple of 2013?

6. **[1979 A1]** Find the set of positive integers with sum 1979 and maximum possible product.

   Can you also answer this question for 2013?

Challenge.

1. Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \qquad \text{for } n \geq 3.$$

   (a) Prove that for any $n \geq 1$,

$$F_n^2 = F_{n+1}F_{n-1} + (-1)^{n-1}.$$

   (b) Prove that for any integer k there are infinitely many n such that F_n is a multiple of k.

   (c) Prove that F_{2013} is a multiple of F_{61}.