Problem Solving Seminar - Fall 2013
Oct. 23

1. An integer *lattice point* is a point such that all coordinates are integers.
   (a) What is the area of the triangle with vertices $(0,0),(5,0)$ and $(8,3)$?
   (b) Let T_0 be the triangle with vertices $(0,0),(2,4)$ and $(5,1)$. Calculate the area of T_0.
   (c) If a,b,c,d are positive integers, what is the area of the triangle with vertices $(0,0),(a,b)$ and (c,d)?

2. In this problem you will consider polygons whose vertices are lattice points. Pick's Theorem states that if such a polygon contains n lattice points in its interior and m lattice points on its boundaries, then its area is $n+\frac{m}{2}-1$.
   (a) Verify that Pick's Theorem holds for the triangle T_0 from Problem 1.
   (b) If a is a positive integer, show that Pick's Theorem holds for any rectangle with vertices of the form $(a,1),(a+4,1),(a,3)$ and $(a+4,3)$.
   (c) Prove Pick's Theorem for any triangle with vertices $(0,0),(1,0),(a,b)$, where a and b are positive integers.

(*Challenge*): Prove Pick's Theorem in the general case.

3. Given a triangle ABC, construct three equilateral triangles ABX, ACY and BCZ that extend externally from the edges. Prove that $AZ=BY=CX$.

4. (a) Suppose that A,B,C,D are points in the plane such that the triangles ABC, ABD, ACD, and BCD each have area at most 1. What is the largest possible area of the quadrilateral $ABCD$?
   (b) Suppose that $P_1,\ldots,P_n$ are a collection of points in the plane such that the triangle formed by any three of them has area at most 1. Prove that there is a triangle T with area at most 4 such that all of the P_i lie inside T or on its boundary.
   (c) Find a set of $P_1,\ldots,P_n$ such that any possible choice of T has area exactly 4.

5. **[1991 A1]** A 2×3 rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$ and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is $(1,1)$.

6. **[1991 A4]** Does there exist an infinite sequence of closed discs $D_1,D_2,\ldots$ in the plane, with centers $C_1,C_2,\ldots$ such that
   (a) the set of C_i have no finite limit point in the plane,
   (b) the sum of the areas of the D_i is finite,
   (c) every line in the plane intersects at least one of the D_i?

Challenge.

1. (a) Suppose that the vertices of a triangle $T = ABC$ are lattice points in the plane, and T contains no other lattice points in either its interior or boundary. What is the largest possible area of T?

   (b) Now consider a triangle $T = ABC$ whose vertices are lattice points in three dimensions, and T contains no other lattice points in either its interior or boundary. Show that the area of T can be arbitrarily large.

   (c) Suppose that the vertices of a quadrilateral $Q = ABCD$ are lattice points in three dimensions, and Q contains no other lattice points in its interior or boundary. What is the largest possible volume of Q?