MATH 7290 Homework 1 - Spring 2013

Due Tuesday, Jan. 29 at 1:30

The notation "Koblitz A.B.C" means Problem C from Chapter A, Section B of the textbook.

[Tues., Jan. 15]

1. (a) Fill in the details of the following outline of a proof that the circle $x^2 + y^2 = 1$ has infinitely many rational points. First, note that $P = (1, 0)$ is a point on the circle. Now consider a line with rational slope s that passes through P. Find the other intersection of the line and circle, and verify that the coordinates are rational. Does this procedure account for all possible rational points on the circle? (See Koblitz p. $1-2$).
   (b) Are there infinitely many rational points on $x^2 + y^2 = 5$?
   (c) Are there infinitely many rational points on $x^2 + 2x + y^2 - 4y + 2 = 0$?
   (d) Are there any circles in the real plane with finitely many rational points?
   Hint: The answer is no if all of the coefficients in the defining equation are rational....

[Thurs., Jan. 17]

2. Koblitz 1.1.5.
3. Koblitz 1.2.2.

[Tues., Jan. 22]

4. The following equation (due to Fermat) defines an elliptic curve:

$$X^3 + Y^3 = 1.$$

   Find a sequence of birational maps that convert it to Weierstrass form $y^2 = x^3 + ax + b$, where a and b are both integers.

5. (See Koblitz 1.3.2 and 1.3.6(a)). In this problem you will show that points and lines are dual in the projective plane $\mathbb{P}^2(K)$.
   (a) Show that if at least one of a, b, and c are non-zero, then

$$ax + by + cz = 0$$

   is the projective completion of an affine line.
   (b) Let $L(a, b, c) := \left\{(x, y, z) \in K^3 - \{0\} \mid ax + by + cz = 0\right\}$. Show that $L(a, b, c)$ is a well-defined injective map from $\mathbb{P}^2(K)$ into the set of projective lines.
   (c) Complete the proof by showing that $L(a, b, c)$ is also surjective.

(d) If $P_i = [x_i : y_i : z_i]$ for $i = 1, 2$, find an equation for the line through these two points, as well as the corresponding dual point $[a : b : c]$.

[Thurs., Jan. 24]

6. (a) Describe the projective completion of the circle $x^2 + y^2 = 1$ over the real field.
   (b) Describe the projective completion of the hyperbola $x^2 - y^2 = 1$ over the real field.

7. Koblitz 1.3.7.