MATH 7290 Homework 4 & 5 - Spring 2013

Due Tuesday, Apr. 16 at 1:30

The notation "Koblitz A.B.C" means Problem C from Chapter A, Section B of the textbook.

[Tues., Mar. 5]

1. Koblitz 2.2.5.
2. Koblitz 2.2.8(a); try to show that n is a square in $\mathbb{F}_{q^{2r}}$.

[Thurs., Mar. 7]

3. Koblitz 2.2.21.

(*Suggested*). Koblitz 2.2.10 – 2.2.17; these problems provide a proof of the "Hasse-Davenport relation", which describes p-adic lifting for Gauss sums.

[Tues., Mar. 12]

4. Koblitz 2.6.1.

(*Optional*). Consider working on this problem if you have some background in analytic number theory. The heuristic argument for the Birch-Swinnerton-Dyer Conjecture described in Section 2.6 ends with the approximation

$$L(E,1) \approx \prod_{p \text{ prime}} \frac{1}{1+p^{-\frac{1}{2}}}$$

for elliptic curves E that have many rational points. Prove that the product

$$\prod_{p \text{ prime}} \frac{1}{1+p^{-\frac{1}{2}}} = 0.$$

5. Koblitz 2.4.4, with part (c) optional. Note that a nontrival character always has modulus $N > 1$. Interestingly, the theta function $\vartheta(t)$ can be viewed as a degenerate case corresponding to the trivial character $\chi = 1$, but the analysis is actually more difficult in this case due to convergence issues.

[Thurs., Mar. 14]

6. Prove that the linear fractional transformations

$$\gamma(\tau) := \frac{a\tau+b}{c\tau+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

give a group action of $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. In other words, show that if $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$, then

$$(\gamma_1\gamma_2)(\tau) = \gamma_1\big(\gamma_2(\tau)\big).$$

7. Write the matrix $\begin{pmatrix} 11 & 41 \\ 15 & 56 \end{pmatrix}$ as a product of $S := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$, which generate Γ.

[Tues., Mar. 19]

8. (a) Following the standard notation for the generators of Γ, verify that $U^3 = -I$, where $U = ST$.
   (b) Using the fact that Γ is generated by S and T, verify that Γ is also generated by S and U.

   (*Optional*) Prove that $\overline{\Gamma} := \Gamma/\{\pm I\}$ is freely generated by S and U. This requires showing that there are no algebraic relations between the order 2 matrix S and order 3 matrix U.

9. Koblitz 3.2.1.

[Thurs., Mar. 21]

10. In this problem you will justify the "Partial Residue Theorem", which was used in the proof of the Valence Formula. Suppose that C is the counterclockwise portion of a circle from $\alpha = e^{2\pi i\theta_1}$ to $\beta = e^{2\pi i\theta_2}$, with the arguments chosen such that $\theta_1 \leq \theta_2$. Prove that
$$\frac{1}{2\pi i}\int_C \frac{dz}{z} = \theta_2 - \theta_1.$$

11. Koblitz 3.2.4.

[Tues., Mar. 26]

12. Write the modular function $f(\tau) := \dfrac{10E_{28}(\tau) + 14E_4(\tau)^4E_{12}(\tau)}{E_4(\tau)\Delta(\tau)^2}$ as a rational function of $j(\tau)$.

[Thurs., Mar. 28]

13. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, prove that
$$E_2(\gamma(\tau)) = (c\tau + d)^2E_2(\tau) - \frac{6ic(c\tau + d)}{\pi}.$$
Hint: Use the transformations for $\gamma = S, T$ and an inductive argument based on the fact that these generate Γ.

14. In general, the derivative of a modular form $f(\tau) \in M_k$ is not modular. In this problem you will explore a modified differential operator that preserves modularity (see also Koblitz 3.2.7).

(a) If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, prove that

$$\frac{d}{d\tau}\gamma(\tau) = \frac{1}{(c\tau + d)^2}.$$

(b) Suppose that $f \in M_k$. Prove that

$$f'(\gamma(\tau)) = (c\tau + d)^{k+2} f'(\tau) + kc(c\tau + d)^{k+1} f(\tau).$$

Hint: Consult equation (2.26) in Koblitz.

(c) Define the "Ramanujan differential operator"

$$\Theta_k := \frac{1}{2\pi i}\frac{d}{d\tau} - \frac{k}{12}E_2(\tau).$$

Prove that $\Theta_k : M_k \to M_{k+2}$. What can you conclude about $\Theta_k(f)$ if $f \in S_k$?

15. Koblitz 3.2.8. Rewrite the identities using Ramanujan operators.