MATH 7290 Homework 6 - Spring 2013

Due Thursday, May 2 at 1:30

The notation "Koblitz A.B.C" means Problem C from Chapter A, Section B of the textbook.

[Tues., Apr. 9]

1. Recall from lecture the Jacobi Triple Product:

$$\prod_{n\geq 1}\left(1-zq^{n-1}\right)\left(1-z^{-1}q^{n}\right)\left(1-q^{n}\right)=\sum_{k\in\mathbb{Z}}(-z)^{k}q^{\frac{k(k-1)}{2}}.$$

This can be proven analytically, though it requires some familiarity with combinatorial power series. In this problem you will prove an auxiliary identity that is the first step in the proof of the Triple Product formula.

In particular, you will show that

$$\prod_{n\geq 1}\left(1+zq^{n}\right)=\sum_{m\geq 0}\frac{z^{m}q^{\frac{m(m+1)}{2}}}{\prod_{j=1}^{m}(1-q^{j})}.$$

(a) Denote the left-hand side by $F(z;q)$. Find a simple functional equation that relates $F(zq;q)$ to $F(z;q)$.

(b) Assuming that the convergence is sufficiently well-behaved so that reordering is legal, denote the series expansion in powers of z by $F(z;q)=:\sum_{m\geq 0}A_m(q)x^m$. What is $A_0(q)$? Plug in the series expansion to part (a) and show that

$$A_m(q)=\frac{A_{m-1}(q)q^m}{1+q^m}.$$

(c) The above recursive formula is easily solvable – do this and find an explicit formula for $A_m(q)$.

(*Optional:*) Try to explain the identity combinatorially by interpreting the left-hand side as the generating function for integer partitions into distinct parts; in other words, the coefficient of $z^m q^\ell$ counts the number of distinct ways of writing

$$\ell=\lambda_1+\cdots+\lambda_k,$$

where $\lambda_1>\lambda_2>\cdots>\lambda_k\geq 1$ are integers.

Remark. In fact, the general theory of "q-difference equation" implies that $F(z;q)$ is the unique solution (this is essentially equivalent to partial differential equations with boundary conditions).

[Thurs., Apr. 11]

2. In the double-coset definition of Hecke operators, one effectively takes the trace over the set of integer matrices with determinant m modulo $\mathrm{SL}_2(\mathbb{Z})$. This eventually leads to the formula

$$f(\tau)\Big|_k T_m = m^{\frac{k}{2}-1} \sum_{\substack{a,d\in\mathbb{N}\\ ad=m}} \sum_{b=0}^{d-1} f(\tau)\Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Show that this is equivalent to the formula we derived using lattice functions (at a minimum, do the case where $m = p$ is prime).

[Tues., Apr. 16]

3. In lecture we used a geometric argument to show that the measure $\frac{dxdy}{y^2}$ is invariant under the action of Γ, where $\tau = x + iy$. Specifically, we claimed that if $\tau \to u(\tau)$, then $dxdy \to |u'(\tau)|^2 dxdy$. In this problem you will verify this fact analytically.

   (a) Write $u(\tau) = v(x, y) + iw(x, y)$, where v and w are analytic functions. What do the Cauchy-Riemann equations say about the partial derivatives v_x, v_y, w_x, and w_y (here a_b means $\frac{da}{db}$)?

   (b) Recall from multivariable calculus that the area differential satisfies

   $$dvdw = \left|\frac{\partial(v, w)}{\partial(x, y)}\right| dxdy,$$

   where the Wronskian is

   $$\left|\frac{\partial(v, w)}{\partial(x, y)}\right| := \det\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}.$$

   Verify that the Wronskian does indeed simplify to $|u'(\tau)|^2$.

4. Recall that the U_m operator can be written in terms of slash operators:

$$f(\tau)\Big|_k U_m = \frac{1}{m} \sum_{j=0}^{m-1} f(\tau)\Big|_k \begin{pmatrix} 1 & \frac{j}{m} \\ 0 & 1 \end{pmatrix}.$$

Prove that if $f(\tau) \in M_k$, then $f(\tau)\big|_k U_m \in M_k(\Gamma_0(m))$.

[Thurs., Apr. 18]

5. Verify that the measure of the fundamental domain $\mathcal{F}_\Gamma$ is

$$\int_{\mathcal{F}_\Gamma} \frac{dxdy}{y^2} = \frac{\pi}{3}$$

by integrating along vertical strips $\sqrt{1 - x^2} \le y \le \infty$.

6. Koblitz 3.1.1.

[Tues., Apr. 23]

7. In this sequence of problems you will determine the relative indices for the congruence subgroups $\Gamma(N), \Gamma_1(N)$ and $\Gamma_0(N)$.

   Work Koblitz 3.2.2 – 3.2.4 and 3.2.7 in the case of prime power N.

   (a) (*Optional:*) Complete Koblitz 3.2.5 – 3.2.7 in the general case.

[Thurs., Apr. 25]

8. The cusps of a congruence subgroup Γ' are the equivalence classes of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ'. In this problem you will consider the case $\Gamma' = \Gamma_0(p)$ for a prime p.

   (a) If s is a multiple of p, prove that $\frac{1}{s}$ is equivalent to the cusp $i\infty$.

   (b) If s is not a multiple of p, prove that $\frac{1}{s}$ is equivalent to the cusp 0.

   (c) Prove that $[i\infty]$ and $[0]$ are the only two cusps; i.e., any $\frac{r}{s}$ is equivalent to one of them.

9. (a) Koblitz 3.1.14.

   (b) Sketch a fundamental domain for $\Gamma_0(p)$ if p is prime.

[Tues., Apr. 30]

(*Optional:*) Koblitz 3.3.10(a)(b)(d).