Problem Solving Seminar - Fall 2014
Aug. 27

1. (a) The sum of the first n natural numbers is given by the well-known formula

$$S_1(n) := \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

Give at least one proof for this fact: induction, Gauss' grouping trick, Linear recurrences, combinatorial arguments, calculus, ...

(b) Similarly, there is a formula for the sum of the first n squares of the form

$$S_2(n) := \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d$$

for some constants a, b, c, d. Use an integral approximation to justify the guess $a = \frac{1}{3}$. Then use any method to find the exact formula.

(c) Do the same for the sum of cubes

$$S_3(n) := \sum_{i=1}^{n} i^3.$$

2. (a) Consider a checkerboard, which is divided into a grid of 8×8 smaller squares. How many squares of *any* size are contained in the gridlines of the board?

(b) How many **rectangles** of any size are contained in the board? Note that this should be at least as large as part (a)!

(c) Can you find a general formula for the squares or rectangles on an $n \times n$ board? How do your formulas compare to Problem 1?

3. [Gelca-Andreescu **19**] Suppose that $a_1, a_2, \ldots, a_n$ are integers arranged in a circle such that $a_1 + a_2 + \cdots + a_n = 1$. Prove that there is a unique starting point a_k such that all of the cyclic partial sums

$$a_k, \quad a_k + a_{k+1}, \quad \ldots, \quad a_k + a_{k+1} + \cdots + a_n, \quad a_k + a_{k+1} + \cdots + a_n + a_1, \ \ldots$$

are positive.

4. Prove that there are two points on the Earth's equator that are exactly the same temperature.

Hint: Let $T(x)$ denote the temperature at radian angle x, and suppose that $T(x+\pi) - T(x) > 0$. What happens as you rotate further around the equator from this point?

5. [**1988 B2**] Prove or disprove: If x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.