LSU Problem Solving Seminar - Fall 2014
Oct. 29

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This week's problem sheet provides strategies for the Virginia Tech Regional Math Contest, which was given last Saturday. Each of the VTRMC problems is preceded by a related, easier problem.

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1. (a) Use the "telescoping" identity $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ to evaluate the sum
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$
   (b) Find constants a, b, c such that $\frac{1}{n(n+1)(n+2)} = \frac{a}{n} + \frac{b}{n+1} + \frac{c}{n+2}$. Use this to evaluate the sum
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$
   (c) Show that $x^4 + a^2x^2 + a^4$ factors into two quadratic polynomials.
   Hint: Start from $x^6 - a^6$ and factor it in two different ways, first a difference of squares, and then as a difference of cubes.

2. [VTRMC **2014 #1**] Find
$$\sum_{n=2}^{\infty} \frac{n^2 - 2n - 4}{n^4 + 4n^2 + 16}.$$

3. (a) Suppose that $f(x)$ is a function such that $f(x) \neq -f(-x)$ at any point. Let
$$I := \int_{-1}^{1} \frac{f(x)}{(f(x) + f(-x))}\, dx.$$
   Make the substitution $x = -u$ and conclude that $I = 1$.
   (b) [Putnam **1987 B1**] Evaluate
$$\int_{2}^{4} \frac{(\ln(9-x))^{1/2}}{(\ln(9-x))^{1/2} + (\ln(x+3))^{1/2}}\, dx.$$

4. [VTRMC **2014 #2**] Evaluate
$$\int_0^2 \frac{(16 - x^2)x}{16 - x^2 + \sqrt{(4-x)(4+x)(12+x^2)}}\, dx.$$

5. (a) Show that if n is an odd integer, then $n^2 - 1$ is a multiple of 8. Iterate this argument to conclude that $n^{2^k} - 1$ is a multiple of 2^{k+2} for any $k \geq 1$.
   (b) Show that if $n^2 - 1$ is a multiple of 8 but not a multiple of 16, then $n^4 - 1$ is a multiple of 16 but not of 32.

6. [VTRMC **2014 #3**] Find the least positive integer n such that 2^{2014} divides $19^n - 1$.

7. (a) Consider an 8×8 chessboard, where adjacent squares alternate in color between Black and white. Show that the upper-left and lower-right squares are the same color. Conclude that if these two squares are removed, then it is impossible to cover the remaining 62 squares with 1×2 and 2×1 rectangles.

   (b) Let n be odd. If the upper-left corner is removed from a $n \times n$ board, show the remaining tiles can be covered with 1×2 and 2×1 rectangles.

   (c) Consider a 10×10 board whose squares are colored in four colors: Black (B), White (W), Red (R), Green (G). The first row is colored with the repeating sequence BWRG, the second row WRGB, the third row RGBW, and the fourth row GBWR, with the fifth row repeating the first, and so on. Count the number of squares of each color and conclude that the board **cannot** be tiled by 1×4 and 4×1 rectangles.

8. [VTRMC **2014 #4**] Suppose that a 19×19 chessboard has the center square removed. Is it possible to tile the remaining 360 squares with 4×1 and 1×4 rectangles?

9. (a) Prove that n and $n+1$ have no common divisor (i.e., an integer $d \geq 2$ such that both n and $n+1$ are multiples of d).

   (b) Prove that if $r \geq 2$, there are no consecutive r-th powers; in other words, there is no m and ℓ such that $m^r = \ell^r + 1$.

   (c) Conclude that there are no integer solutions to $n(n+1) = m^r$ with $r \geq 2$.

10. [VTRMC **2014 #5**] Let $n \geq 1$ and $r \geq 2$ be positive integers. Prove that there is no integer m such that $n(n+1)(n+2) = m^r$.

11. (a) Let A and B be finite sets of integers. Show that there is a unique pair $a \in A$ and $b \in B$ such that $a + b$ is maximal.

    (b) Show that the set of integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $ad - bc = 1$ is a *group*. In other words, if A and B are two such matrices, then so is AB and A^{-1}.

12. [VTRMC **2014 #6**] (Modified and simplified)

    (a) Let G be a group, so that it is closed under multiplication and inverses. Let A and B be subsets of G. Show that if $a'b' = ab$ for some $a, a' \in A$ and $b, b' \in B$, then there is an element $g \in G$ such that

    $$a'b = abg \quad \text{and} \quad ab' = abg^{-1}.$$

    (b) Suppose that M is a matrix of finite order, so that $M^n = I$ (the identity matrix) for some integer n. Prove that there is no function f that is defined on powers of M such that $f(xy) > f(x)$ or $f(xy^{-1}) > f(x)$.

13. For integers $n, k \geq 0$, the *binomial coefficient*

    $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

    counts the number of distinct ways of choosing k objects from a set of n

    (a) Consider a string of k E's and ℓ S's, for example EESESSS (if $k = 3, \ell = 4$). Explain why the total number of distinct strings is $\binom{k+\ell}{\ell}$.

    (b) Show that if $k < \frac{n}{2}$, then

    $$\binom{n}{k} < \binom{n}{k+1}.$$

(c) Prove that if $m < n$, then

$$\binom{m}{k} \cdot \binom{n}{\ell} < \binom{m+1}{k} \cdot \binom{n-1}{\ell}.$$

14. [VTRMC **2014 #7**] (Modified and simplified) Given two integer points $A = (x_1, y_1), B = (x_2, y_2)$, define $d(A, B)$ to be the number of South-East paths from A to B. (Note that if A is not northwest of B, then $d(A, B)$ can still be defined, and is simply zero).

    (a) Find $d(A, B)$ as a function of the coordinates of A and B.

    (b) Suppose that A_1 is northeast of A_2 and B_1 is northeast of B_2. Prove that

    $$d(A_1, B_1)d(A_2, B_2) > d(A_1, B_2)d(A_2, B_1).$$

Answers for numerical problems. **#1**: $\frac{1}{14}$, **#2**: 1, **#3**: 2^{2012}, **#4**: No.

Challenge.

1. Recall Problem 8. Let n be an odd integer, and consider an $n \times n$ board.

    (a) Prove that if the center square is removed, then it **is** possible to tile the rest of the board with 1×4 and 4×1 rectangles if $n = 1, 7, 9, 15, 17, \ldots$ (remainder 1 or 7 when divided by 8).

    (b) Prove that if the center square is removed, then it **is not** possible to tile the rest of the board with 1×4 and 4×1 rectangles if $n = 3, 5, 11, 13, \ldots$ (remainder 3 or 5 when divided by 8).

    (c) Prove that if n has remainder 3 or 5 when divided by 8, there is some square (not the center!) that can be removed such that the remainder of the board can be covered with 1×4 and 4×1 rectangles.