LSU Problem Solving Seminar - Fall 2014
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Useful facts:

- **Arithmetic-Geometric Mean Inequality.** If $a_1, \dots, a_n$ are non-negative real numbers, then
$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$
Furthermore, the right side is strictly larger than the left unless all of the a_i are equal.

- **Cauchy-Schwarz Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\left(a_1 b_1 + \cdots + a_n b_n\right)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right).$$
Furthermore, the right side is strictly larger unless $(b_1, \dots, b_n) = (\lambda a_1, \dots, \lambda a_n)$ for some real λ.

- **Triangle Inequality.** If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are real numbers, then
$$\sqrt{(a_1 + b_1)^2 + \cdots + (a_n + b_n)^2} \leq \sqrt{a_1^2 + \cdots + a_n^2} + \sqrt{b_1^2 + \cdots + b_n^2}.$$
Written in vector notation and Euclidean distance, $|\overrightarrow{a} + \overrightarrow{b}| \leq |\overrightarrow{a}| + |\overrightarrow{b}|$.

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1. For each of the following pairs, determine which expression is larger (without using a calculator!):
   (a) 10^e or e^{10}?
   (b) $\sqrt[2014]{2014}$ or $\sqrt[2013]{2013}$?
   (c) $2014!$ or 1007^{2014}?
   (d) $\sqrt{2014} - \sqrt{14}$ or $\sqrt{2013} - \sqrt{13}$?

2. (a) Find the minimum value over all positive x of the expression $\dfrac{x}{2} + \dfrac{18}{x}$.
   (b) Find the minimum value over all positive x of the expression $x^2 + x - 1 + \dfrac{1}{x^3}$.
   Hint: Instead of using calculus, try to apply one of the listed inequalities.

3. You are asked to specify the dimensions of a box with volume V in order to minimize shipping and materials costs. The dimensions are height h, length ℓ, and width w.
   (a) The postal service generally sets prices based on the *linear length* of a box, which is the sum of the three dimensions $h + \ell + w$. What is the minimum linear length that you can achieve? What is the numerical value if the volume is 1000 cubic inches?
   (b) The construction of the box will require the fewest materials if the surface area is as small as possible. What is the minimum achievable surface area? What is the numerical value for a volume of 1000 cubic inches?
   (c) Using the same amount of material as in the 1000 cubic inch box from part (b), now find the dimensions of a box without a lid that contains the largest possible volume.

4. (a) Determine which is larger:

$$\left(1+2+\cdots+n\right)^2 \quad \text{or} \quad n\cdot\left(1^2+2^2+\cdots+n^2\right)?$$

(b) In general, if $x_1,\ldots,x_n$ are real numbers that are not all equal, determine which is larger:

$$\left(x_1+x_2+\cdots+x_n\right)^2 \quad \text{or} \quad n\cdot\left(x_1^2+x_2^2+\cdots+x_n^2\right)?$$

5. [Gelca-Andreescu **94**] Find

$$\min_{a,b\in\mathbb{R}} \max\left\{a^2+b,\ a+b^2\right\}.$$

Hint: Split the max into two inequalities.

6. [VTRMC **2002 #7**] Let $\{a_n\}_{n\geq 1}$ be an infinite sequence with $a_n \geq 0$ for all n. For $n\geq 1$, let b_n denote the geometric mean of $a_1,\ldots,a_n$, that is, $(a_1\cdots a_n)^{1/n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

7. [Putnam **1975 B6**] Let $h_n = \sum_{r=1}^{n} \frac{1}{r}$. Show that

$$n - \frac{n-1}{n^{1/(n-1)}} > h_n > n(n+1)^{\frac{1}{n}} - n$$

for $n \geq 2$.

Challenge.

1. (a) Prove that the following inequality holds for all non-negative x, y, z:

$$6x^2y^2z^2 \leq x^3y^2z + x^3yz^2 + x^2y^3z + x^2yz^3 + xy^3z^2 + xy^2z^3.$$

(b) Prove that the following inequality holds for all non-negative x, y, z:

$$x^2yz + xy^2z + xyz^2 \leq x^4 + y^4 + z^4.$$