Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 26 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Dec. 3.
- Putnam Mathematical Competition, **Sat., Dec. 6.** The Exam will take place in Lockett 232 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2014
Nov. 12

Prof. Karl Mahlburg

Useful facts:

- **Binomial Coefficients.** Given two non-negative integers n and k, the number of ways of choosing k (unordered) objects from a set of n is $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ (this is read as "n choose k"). They satisfy the recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.
- **Binomial Theorem.** For an integer $n \geq 0$, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.
- **Number of subsets.** There are 2^n distinct subsets of a set with n elements.

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1. The *Fibonacci sequence* are defined by $F_0 = F_1 = 1$, and $F_{n+1} := F_n + F_{n-1}$ for $n \geq 1$.
   (a) Calculate the first several terms in the sequence.
   (b) Let $s(n)$ denote the number of sequences of 1s and 2s that sum to n. For example, for $n = 4$ the sequences are
   $$22,\ 211,\ 121,\ 112,\ 1111,$$
   so $s(4) = 5$. Find expression for $s(n)$ in terms of the Fibonacci numbers.
   (c) Find (and prove) a formula for the sum of the first $n+1$ terms,
   $$F_0 + F_1 + \cdots + F_n.$$
   (d) Prove that
   $$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}.$$
2. Recall that any two distinct points in the plane determine a line, and that any two distinct lines either intersect in a point or are parallel. In this problem you will consider a collection of points $\{P_1, \ldots, P_n\}$ and the set of lines that they determine; this means all of the lines given by any pair P_i, P_j.
   (a) If $P_1, P_2, \ldots, P_{10}$ are distinct points in the plane, what is the maximum number of lines that they determine? What is the minimum number of lines?
   (b) If $L_1, L_2, \ldots, L_{10}$ are distinct lines in the plane, what is the maximum number of intersection points? What is the minimum number of points?
   (c) Prove that if $P_1, \ldots, P_n$ are distinct points that do not all lie on the same line, then they determine at least n lines.

(d) Prove that if $L_1, \dots, L_n$ are distinct lines that intersect in more than 1 point, then they intersect in at least $n-1$ points.

3. (a) Evaluate $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$ and $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$.

   Hint: For the second sum, take the derivative of the Binomial Theorem.

   (b) Given n objects, you may choose an arbitrary subset from them. Prove that there are an equal number of ways to choose an even number of objects as an odd number.

   (c) [Gelca-Andreescu **869**] Prove the identity $\displaystyle\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$.

   Try to prove this in several ways: 1. Induction, 2. Generating functions, 3. Counting arguments, ...

   (d) [Gelca-Andreescu **870**] If $0 \le m \le n$, evaluate $\displaystyle\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^m\binom{n}{m}$.

4. (a) Consider $n \times n$ matrices whose entries are 0s or 1s. How many such matrices are possible?

   (b) How many of the matrices have the property that the sum of each row is odd?

   (c) How many of the matrices have odd row **and** column sums?

5. [VTRMC **2005 #3** (Modified)] We wish to tile a $n \times 1$ grid with squares (1×1) and dominoes (2×1). Each position in the grid must be covered by either a square or a domino. Furthermore, additional square tiles may be stacked on top, up to a height of two, but at most one square can be stacked on any domino. Let $t(n)$ be the number of ways the strip can be tiled according to the above rules; for example, $t(1) = 2$ and $t(2) = 7$. Calculate $t(6)$.

6. [Putnam **2005 A2**] Let $\mathbf{S} = \{(a,b) | a = 1, 2, \dots, n, b = 1, 2, 3\}$. A *rook tour* of $\mathbf{S}$ is a polygonal path made up of line segments connecting points $p_1, p_2, \dots, p_{3n}$ in sequence such that

   (i) $p_i \in \mathbf{S}$,

   (ii) p_i and p_{i+1} are a unit distance apart, for $1 \le i < 3n$,

   (iii) for each $p \in \mathbf{S}$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1,1)$ and end at $(n,1)$?

   For example, the following is a rook tour for $n = 5$:

$$(1,1), (2,1), (2,2), (1,2), (1,3), (2,3), (3,3), (4,3),$$
$$(5,3), (5,2), (4,2), (3,2), (3,1), (4,1), (5,1).$$

Challenge.

1. Refer to Problem 2. Suppose that $P_1, \dots, P_n$ are distinct points such that at most $n-2$ are contained in the same line. What is the minimum number of lines determined by the points?