Important upcoming dates:

- The Problem-Solving Seminar will **not** meet on Wednesday, Nov. 26 due to the Thanksgiving holiday. The last meeting of the semester will be Wednesday, Dec. 3.
- Putnam Mathematical Competition, **Sat., Dec. 6.** The Exam will take place in Lockett 232 from 8:30 A.M. – 5:00 P.M.

LSU Problem Solving Seminar - Fall 2014
Nov. 19

Prof. Karl Mahlburg

1. (a) Simplify the following expressions to rational fractions of the form $\frac{a}{b}$:

$$1+\cfrac{1}{1+\cfrac{1}{1}}, \qquad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}, \qquad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}.$$

How are these fractions related to the Fibonacci numbers?

(b) Let s_n denote the expression

$$1+\cfrac{1}{1+\cdots\cfrac{1}{1+\cfrac{1}{1}}}$$

that has exactly n 1s reading along the lower diagonal. For example, the expressions in part (a) are s_3, s_4, and s_5. Determine $\lim_{n\to\infty} s_n$.

Hint: Find a recurrence that relates s_{n+1} and s_n.

2. (a) Define a sequence by $a_0 = 1$ and for $n \geq 1$, by the recurrence

$$a_{n+1} = 1 + \frac{2}{a_n}.$$

Calculate the first several terms of the sequence. Determine $\lim_{n\to\infty} a_n$.

Hint: Denote the limit by a. Letting $n \to \infty$ in the recurrence implies that a must satisfy $a = 1 + \frac{2}{a}$.

(b) Define a sequence by $b_0 = 1, b_1 = 3$ and for $n \geq 1$, by the recurrence

$$b_{n+1} = 2b_n - b_{n-1}.$$

Show that this sequence diverges to infinity. Determine $\lim_{n\to\infty} \frac{b_{n+1}}{b_n}$.

(c) Define a sequence by $c_0 = 1, c_1 = 3$ and for $n \geq 1$, by the recurrence

$$c_{n+1} = c_n + \frac{2}{c_{n-1}}.$$

Does this sequence converge to a finite limit? Determine $\lim_{n\to\infty} c_n$.

3. (a) Evaluate $\sqrt{2\sqrt{2\sqrt{2\sqrt{\cdots}}}}$.

   Hint: If x is the (limiting) value of this expression, then $x = \sqrt{2 \cdot x}$.

   (b) Evaluate $\sqrt{4+3\sqrt{4+3\sqrt{4+3\sqrt{\cdots}}}}$.

   (c) Evaluate $\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{\cdots}}}}$.

4. A pocket calculator has the ability to add, subtract, and take reciprocals, but cannot multiply or divide. Find a way to use the available operations in order to multiply two rational numbers. (In particular, do not just perform multiplication by successive addition, which is rather difficult – and tedious – when multiplying fractions). *Hint: As a first step, try to calculate the square of a given number.*

5. [Gelca-Andreescu **339**] Show that the sequence

$$\sqrt{7}, \quad \sqrt{7-\sqrt{7}}, \quad \sqrt{7-\sqrt{7+\sqrt{7}}}, \quad \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}$$

   converges, and evaluate its limit.

6. (a) Define a sequence d_n by $d_0 = 1, d_1 = 2, d_2 = 4$ and

$$d_{n+1} = \frac{d_n d_{n-1}}{d_{n-2}}$$

   for $n \geq 2$. Find a simple formula for d_n.

   (b) [VTRMC **1983 #8**] A sequence f_n is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}}$$

   for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that f_n is integer-valued for all $n \geq 0$.

7. [Putnam **1966 A6**] Let

$$a_n := \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots+(n-1)\sqrt{1+n}}}}}.$$

   Prove that $\lim_{n\to\infty} a_n = 3$.

Challenge.

1. (a) It is a surprising fact that the following expression converges to a finite limit:

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}.$$

   Determine its value.

   (b) Now consider the general expression

$$x^{x^{x^{\cdots}}}.$$

   Show that if $x = 2$, the expression diverges to ∞. For which values of x does this have a finite limit?