LSU Problem Solving Seminar - Fall 2014
Dec. 3

Prof. Karl Mahlburg

Putnam Mathematical Competition, **Sat., Dec. 6**, Lockett Hall 232, 8:30 A.M. – 5:00 P.M.

Test-taking tips:

- **Format.** The Exam is given in two 3-hour sessions of 6 problems each, with a lunch break from 12:00 – 2:00 P.M. The morning session's problems are labeled **A1** – **A6**, and the afternoon's **B1** – **B6**.
- **Grading.** Each problem is graded out of **10** points, for a maximum possible score of 120. Typically there is very little partial credit given, as a submitted problem will receive **0, 1, 2, 9,** or **10** points.
- **A1/A2/B1.** In recent years these three problems have been the "easiest" part of the exam. More generally, the problems in each session are roughly ordered by difficulty. This is not an absolute rule, but you should expect that **A1** will have a relatively short solution, whereas **A6** will not. You should spend at least 10 minutes each on **A1, A2, B1** before moving on to the rest of the Exam.
- **1 hour per write-up.** In order to get full credit, your solutions must be written very carefully. If you use a result from a course, refer to it by name (e.g. Fundamental Theorem of Calculus). After you solve a problem, you should plan on spending approximately one hour writing your solution. Remember, it is better to solve one problem completely than several problems partially.

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1. You are sitting in a completely dark room in front of a table with a regular deck of 52 cards. You are told that all of the Spades are face-up, while all of the other cards are face-down. Find a way to split the deck into two piles so that each contains an equal number of face-up cards (you are allowed to flip cards).

2. The faces of a cube are colored Black or White. How many inequivalent colorings are there? Two colorings are equivalent if one can be rotated to the other; for example, there is only one inequivalent coloring with 1 Black face and 5 White faces.

3. A group of people is waiting outside a room. Once they enter, a Black or White hat will be placed on each person's head, and they will not be allowed to leave until they line up against the wall with all White hats at one end and Black hats on the other. Within the room, the people are not allowed to communicate or touch in any way, and no person can see the hat on his own head. However, they can discuss and agree upon a strategy before entering the room – how can they achieve the necessary goal?

4. (a) An *integer composition* of size n is a sequence of positive integers that sum to n. For example, the compositions of 4 are:

   $$4,\ 3+1,\ 1+3,\ 2+2,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 1+1+1+1.$$

   How many compositions of size n are there?

   (b) [Putnam **2003 A1**] Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers, $n = a_1 + a_2 + \cdots + a_k$, with k an arbitrary positive integer and $a_1 \le a_2 \le \cdots \le a_k \le a_1 + 1$? For example, with $n = 4$ there are four ways: 4, 2+2, 1+1+2, 1+1+1+1.

5. (a) Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots.$$

Hint: Consider the Taylor series of $\ln(1-x)$ *around* $x=0$.

(b) [Putnam **1997 A3**] Evaluate

$$\int_0^\infty \left(x-\frac{x^3}{2}+\frac{x^5}{2\cdot 4}-\frac{x^7}{2\cdot 4\cdot 6}+\cdots\right)\left(1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}+\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\cdots\right)dx.$$

6. (a) Find all real numbers α such that

$$\left(x^2-1\right)^2+(\alpha x)^2=x^4+1.$$

Find all β such that

$$\left(x^3-\beta x\right)^2+\left(\beta x^2-1\right)^2=x^6+1.$$

(b) [Putnam **2007 B4**] Let n be a positive integer. Find the number of pairs P, Q of polynomials with real coefficients such that

$$(P(X))^2+(Q(X))^2=X^{2n}+1$$

and $\deg P > \deg Q$.

7. (a) Let $f(x)$ be a polynomial with integer coefficients. Prove that if m and n are integers with $m-n=d$, then $f(m)-f(n)$ is a multiple of d.

(b) [Putnam **2000 A6**] Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \ldots$ of integers such that $a_0=0$ and $a_{n+1}=f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m=0$ then either $a_1=0$ or $a_2=0$.