

Problem Solving Seminar - Fall 2014
Sep. 3

1. (a) Insert plus signs into the following expression so that the sum is correct:

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 = 99.$$

- (b) Do the same for the reversed sequence 987654321.
 (c) Prove that there is no possible placement of plus signs so that the following sum is correct:

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 = 100.$$

However, there **are** solutions if you use both plus and minus signs – can you find one?

Hint: What is the remainder of both sides when divided by 9?

2. (a) A platoon of soldiers must cross a river that is too deep to ford, and the intended bridge has been destroyed. The commanding officer spies two boys playing in a rowboat, which is so small that it can only contain the two boys or one soldier. Nevertheless, the soldiers manage to successfully cross the river – how?
 (b) A man must take an alligator, a goat, and a watermelon across the river, but his boat is only large enough for himself plus just one of the alligator, goat or watermelon. The alligator will eat the goat if he ever leaves them together on shore, and the goat will eat the watermelon if they are alone. How can the man safely transport all three across the river?
3. Create a grid of 7 squares, and place 3 coins Heads side up on the left, 3 coins Tails side up on the right, with one open space in the middle. Your goal is to reverse the positions of the coins. A move consists of shifting a Head to the right if there is an empty space, or jumping a Head over one Tail into an empty space, with corresponding leftward moves for Tails.

For example, in the simpler case of 4 coins in a 5-square grid, the solution is

Initial	H	H		T	T
Move 1	H		H	T	T
Move 2	H	T	H		T
Move 3	H	T	H	T	
Move 4	H	T		T	H
Move 5		T	H	T	H
Move 6	T		H	T	H
Move 7	T	T	H		H
Move 8	T	T		H	H

How many moves are required for the 1×7 grid?

4. [Gelca-Andreescu **61**] In a communications network all of the connections between servers transmit data in only one direction. Furthermore, the network is arranged so that if data exits any server, then it is impossible for it to ever return to that server. Show that there is some server that only transmits data, and also a server that only receives data.

Hint: Consider the longest possible path in the network.

5. [Putnam **1965 B4**, via Gelca-Andreescu **62**] At a certain party various pairs of boys and girls dance with each other throughout the evening. Assume that there is no boy who dances with every girl, and that every girl dances with at least one boy. Prove that there are two girlboy couples (g, b) and (g', b') that dance such that the mixed pairs (g, b') and (g', b) do not dance.

Challenge.

- (a) Refer to Problem 3. How many moves are required to reverse the positions of n Heads and n Tails in a grid of length $2n + 1$?
- (b) Now consider a different puzzle: Place 8 coins in a grid of length 10 in the order HTHTHTHT, with the final two spaces empty. In this game a move consists of moving any two adjacent coins into the empty space, preserving their order. The goal is to place all of the Heads on the left, and all of the Tails on the right, with the empty spaces in between. Show that the shortest solution requires 8 moves.
- (c) Continuing the previous part, place $2n$ coins in a grid of length $2n + 2$ alternating Heads and Tails. How many moves are required to place all of the Heads on the left and all of the Tails on the right?