

Problem Solving Seminar - Fall 2014
Sep. 10

1. (a) You have a length of 100 feet of fencing with which to construct a garden in the shape of a right-triangle. What is the largest possible area that can be enclosed?
(b) You are now given 200 feet of fencing to enclose a square garden which is closed to the north with a semi-circular space for a gazebo. What is the largest area that can be covered?
2. [Gelca-Andreescu **399**] On Monday, two cars drove from Baton Rouge to Atlanta. Remarkably, although they traveled along two different routes, they were never separated by more than 5 miles throughout the entire trip. On Tuesday, the first car drove back to Baton Rouge. Finally, on Wednesday, the first car drove to Atlanta again, while the second car drove back to Baton Rouge, both taking the same route as on Monday. Is it possible that they were **always** further than 5 miles apart on Wednesday?
3. Suppose that $f(x)$ is a polynomial of degree n that is never negative, so that $f(x) \geq 0$ for all x . Note that this condition is clearly satisfied if $f(x)$ contains only even powers of x and all coefficients are positive.
 - (a) Show that n must be even.
 - (b) Find an example of an $f(x)$ that contains at least one *odd* power of x . Find an example of an $f(x)$ with at least one negative coefficient.
 - (c) Let

$$g(x) := f(x) + f'(x) + f''(x) + \cdots + f^{(n)}(x).$$

Prove that $g(x)$ is never negative.

Hint: Consider the critical points of $g(x)$.

4. [VTRMC **1980 #5**] Show that $e^x < (1+x)^{1+x}$ for all $x > 0$.
5. [Putnam **1980 B1**] For which real numbers c is

$$\frac{e^x + e^{-x}}{2} \leq e^{cx^2}$$

for all x ?

Challenge.

1. (a) You are swimming a circular pool when a hungry lion appears and begins stalking the edge of the water*. The lion is single-minded in his focus on you, and will always adopt the optimal strategy in order to catch you when you exit. He can run exactly 4 times faster than you are able to swim. If you reach the edge of the pool at a point that the lion has not yet reached, then you will be able to quickly stand up and scare him into retreating. Find a strategy that allows you to safely escape.

* Although they can swim, lions do not typically hunt in the water; however, tigers sometimes do!

- (b) Now consider a rectangular pool of length ℓ and width w . For what values of (ℓ, w) will you be able to escape?