- Virginia Tech Mathematics Contest. Sat., Oct. 25. **Sign-up deadline: Oct. 1**.
- Putnam Mathematical Competition. Sat., Dec. 6. **Sign-up deadline: Oct. 8**.

Website: `www.math.lsu.edu/~mahlburg/teaching/2014-Putnam.html`

Problem Solving Seminar - Fall 2014
Sep. 24

Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial with real coefficients. A *root* of f is a value r such that $f(r) = 0$. Useful Facts:

- The **Rational Roots Test** states that if all of the a_i are integers and $r = \frac{p}{q}$ is a root, then p is a divisor of a_0 and q is a divisor of a_n.
- The **Fundamental Theorem of Algebra** states that a polynomial of degree n has exactly n complex roots, counted with multiplicity. In particular, it has at most n real roots. Furthermore, if the roots are $r_1, \ldots, r_n$, then $f(x) = c(x - r_1)\cdots(x - r_n)$ for some constant c.
- **Descartes' Rule of Signs** states that if the non-zero coefficients of $f(x)$ change sign s times, then f has at most s positive roots (and that the actual number of positive roots is less than s by some multiple of 2). Replacing x by $-x$ gives a similar test for negative roots.

1. Find the real roots (with multiplicity) of the following polynomials:
   (a) $x^2 - 4x + 4$,
   (b) $x^3 + 3x^2 - 8x + 4$,
   (c) $3x^4 + 8x^3 + 3x^2 + 16x - 6$.
2. (a) Show that $x^{999} - x^{99} - x^9 - 1$ is divisible by $x^3 + 1$.
   (b) Is $x^{999} + x^{99} + x^9 + x$ divisible by $x^3 + 1$? If not, what is the remainder?
   Hint: if there is a remainder, it is a polynomial of degree less than 3.
3. Show that for every $n \geq 1$, the fractions $\dfrac{n^3 + n^2 + 1}{n + 1}$ and $\dfrac{n^3 + n^2 + 1}{n^2 + 1}$ are not integers (in fact, they are in lowest terms!).
4. Find real coefficients a and b such that
   $$f(x) = x^3 - ax^2 + 11x - b$$
   has roots $n, n + 1, n + 2$ for some integer n.
   Hint: Refer to the Fundamental Theorem of Algebra.
5. (a) [Gelca-Andreescu **399**] Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.
   (b) This is not necessarily true if $P(x)$ has even degree. Show that $P(x) = x^2 + 2x$ provides a counterexample.

6. (a) Find a polynomial $f(x)$ with integer coefficients whose roots are $\frac{1}{2}, -\frac{3}{5}$, and 2.
   (b) Prove that if $f(x)$ is a polynomial with integer coefficients and $f(\frac{2}{3}) = 0$, then $f(x) = (3x - 2) \cdot g(x)$, where $g(x)$ is a polynomial with integer coefficients.

7. [Putnam **2004 B1**] Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Challenge.

1. Let $f(x) = x^{2014} - 2x + 1$.
   (a) Prove that $f(x)$ has two real roots.
   (b) Prove that $f(x)$ has no repeated roots.
   (c) Prove that $f(x)$ has a root $r \in (0, 1)$, and that any other root α (including complex!) satisfies $|\alpha| > r$.