- Virginia Tech Mathematics Contest. Sat., Oct. 25. **Sign-up deadline: Oct. 1**.
- Putnam Mathematical Competition. Sat., Dec. 6. **Sign-up deadline: Oct. 8**.

Website: `www.math.lsu.edu/~mahlburg/teaching/2014-Putnam.html`

Problem Solving Seminar - Fall 2014
Oct. 1

1. Calculate the following antiderivatives (indefinite integrals):

    (a) $\int \cos(2x)\, dx$,

    (b) $\int \frac{1}{x(x+1)}\, dx$,

    (c) $\int \left(x^4 + x^2\right) \sqrt{x^2+2}\, dx$,

    Hint: Write $x^4 + x^2 = x(x^3 + x)$ *and pull the* x *inside the square root.*

2. (a) Evaluate $\int_{-1}^{1} x e^{x^2}\, dx$.

    Hint: You do not need an antiderivative!

    (b) Evaluate $\int_{0}^{1} 2x e^{x^2}\, dx$.

    (c) [Gelca-Andreescu **444**]

    $\int \left(1 + 2x^2\right) e^{x^2}\, dx$.

3. (a) Evaluate the integral

$$\int_0^2 \frac{x^2}{2} + \sqrt{2x}\, dx.$$

    (b) If $b > 0$, evaluate the integral

$$\int_0^b \frac{x^2}{b} + \sqrt{bx}\, dx.$$

    (c) Let $f(x) := \frac{x^2}{b}$. If $x \geq 0$, what is the inverse function $f^{-1}(x)$? (Recall that this function satisfies $f(f^{-1}(x)) = x$).

    (d) Evaluate

$$\int_0^{\sqrt{\frac{\pi}{2}}} \sin\left(x^2\right) + \sqrt{\frac{2}{\pi} \arcsin\left(\sqrt{\frac{2}{\pi}} x\right)}\, dx.$$

    Hint: Show that the integral is of the form $\int_0^b \left(f(x) + f^{-1}(x)\right)\, dx$ *and draw a picture.*

4. (a) Find the general solution $f(x)$ to the differential equation $f' + f = e^{-x}$.

    Hint: Multiply by the integration factor e^x.

    (b) [VTRMC **2007 #3**] Solve the initial value problem $\frac{dy}{dx} = y \ln y + y e^x$, $y(0) = 1$ (i.e., find y as a function of x).

5. Let $f(x) := x - \frac{1}{2}$.

   (a) Show that $\int_0^1 f(x)\, dx = 0$.

   (b) Find a value of $0 < \alpha < 1$ such that

$$\left| \int_0^{\alpha} f(x)\, dx \right| = \frac{1}{8}.$$

6. [Putnam **2007 B2**] Suppose that $f : [0,1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x)dx = 0$. Prove that for every $\alpha \in (0,1)$,

$$\left| \int_0^{\alpha} f(x)\, dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} \left| f'(x) \right|.$$

Challenge.

1. Suppose that $k \geq 1$ is a fixed integer. For any integer $n \geq 1$, let

$$H_k(n) := \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{kn}.$$

   (a) Show that $\frac{k-1}{k} \leq H_k(n) \leq k$ for all n.

   (b) Calculate the limit $\lim_{n \to \infty} H_k(n)$.