Problem Solving Seminar - Fall 2014
Oct. 8

Useful facts from geometry and trigonometry:

- *Law of Cosines.* If a triangle has sides of lengths a, b, and c, and α is the angle opposite the side of length a, then
$$a^2 = b^2 + c^2 - 2bc\cos(\alpha).$$
- *Law of Sines.* If β is the angle opposite b, and γ is the angle opposite c, then
$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c} = \frac{1}{2R},$$
where R is the radius of the circumscribed circle (which contains the vertices of the triangle).
- For all x,
$$\sin^2(x) + \cos^2(x) = 1,$$
- For all x and y,
$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y).$$

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1. (a) What is the area of an isosceles triangle with side lengths $(5, 5, 6)$?
   (b) What is the area of an isosceles triangle with side lengths $(5, 5, 8)$?
   (c) Are there other examples of two triangles with integer side lengths and the same area?
2. (a) Find the radius of the circumscribed circle for an equilateral triangle with side length 3.
   (b) Find the radius of the circumscribed circle for a triangle with side lengths $(2, 3, 4)$.
3. Show that there are (not necessarily convex) polygons $P_3, P_4, P_5, P_6, P_7, P_8$, and P_9 with the following properties:
   - All of the sides have integer lengths **and** are at integer coordinates,
   - The area of P_j is j,
   - The perimeter of P_j is 12.

   Hint: Start by finding P_9 and P_6.
4. Let O be the center of two concentric circles, of radii 4 and 8. A line from O hits the outer circle at the point B, and the inner circle at the point A. There is another point C on the outer circle such that $|AC| = 5$. Find $|BC|$.
5. (a) Find constants a and θ such that
$$\sin(x) + \cos(x) = a\cos(x - \theta).$$
   (b) [Gelca-Andreescu **659**] Find the range of the function $f(x) = \big(\sin(x) + 1\big)\big(\cos(x) + 1\big)$.
6. [VTRMC **2013 #2**] Let ABC be a right triangle with $\angle ABC = 90°$, and let D on AB be such that $|AD| = 2|DB|$. What is the maximum possible value of $\angle ACD$?

7. [Putnam **1978 B1**] Find the area of a convex octagon inscribed in a circle that has four consecutive sides of length 3 and the remaining four sides of length 2. Give the answer in the form $r + s\sqrt{t}$, where r, s, and t are integers.

Challenge.

1. For $n \geq 3$, let R_n be a regular polygon (so all sides are equal) with n vertices inscribed in a circle of radius 1.

   (a) Calculate $A_n :=$ Area of R_n.

   (b) Use similarity of various triangles within the regular pentagon to prove that $\cos(36°) = \frac{1+\sqrt{5}}{4}$. Use this to calculate A_5 as an algebraic expression (i.e., an exact value including fractions and square roots, but no sines or cosines).

   (c) What is $\lim_{n\to\infty} A_n$? Prove your answer.